ELECTRICITY AND MAGNETISM FOR ENGINEERS

PART II

ELECTROSTATICS AND ALTERNATING CURRENTS

ELECTRICITY AND MAGNETISM FOR ENGINEERS

BY

HAROLD PENDER

PART I—ELECTRIC AND MAGNETIC CIRCUITS.
380 pages, 6 × 9, Illustrated.

PART II—ELECTROSTATICS AND ALTERNATING CURRENTS.
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ELECTRICITY AND MAGNETISM

FOR

ENGINEERS

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BY

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PREFACE

In the following pages is given, from an engineering point of view, (1) a description of the more important effects commonly described as electric and magnetic phenomena, (2) a statement of the fundamental laws in accord with which these phenomena have been found to occur, and (3) the application of these laws to some of the simpler problems which arise in connection with the generation, transmission and utilization of electric energy.

Particular emphasis is laid upon exact *quantitative* statements of the fundamental laws or principles. Both safety and economy demand that the engineer be able to answer not only "how," but also "how much." To this end, the student of engineering should be taught to analyze, not only qualitatively, but also *quantitatively*, each problem which may be presented to him.

Most of the simpler formulas used by scientists and engineers are special cases of certain general relations, and these special formulas are applicable only under certain specific conditions. One of the most common causes of confusion on the part of the beginner arises from his attempt to apply such special formulas to cases to which they are not applicable. This is due in part to the failure in many text-books to state the *limitations* of such formulas. Particular care is therefore taken in these pages to state specifically the exact conditions under which each formula is applicable.

The procedure adopted throughout the book is to pass from simple phenomena, known to practically every school-boy, to the more complex phenomena and principles with which the engineer has to deal.

For convenience the book has been divided into two parts. Part I deals with the electric and the magnetic circuits, and Part II with electrostatics and alternating currents. Each part of the book can readily be covered in four hours of classroom work per week for a term. Part I may be looked upon as an introduction to the study of direct-current machinery, and Part II as an introduction to the study of alternating-current machinery.

At the end of each important section are given one or more problems, illustrating the principles developed in the text. The answers to these problems are also given. The student should be required to solve each problem, and when time is available additional problems, without answers, should be assigned. It is only by the solution of numerical problems that the student can understand the full significance of the relations developed in the text.

The two volumes of this book cover substantially the same ground as that of the author's "Principles of Electrical Engineering," McGraw-Hill Book Company, 1911. The method of treatment, however, is distinctly different, the various laws and relations are more fully discussed, and a greater number of practical applications is given.

HAROLD PENDER.

PHILADELPHIA, PA.,
April, 1919.

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ELECTRICITY AND MAGNETISM FOR ENGINEERS

PART II
ELECTROSTATICS AND ALTERNATING CURRENTS

XII

ELECTRIC FIELDS IN DIELECTRICS

136. Electric Fields of Force.—Any region in which a particle of electricity is acted upon by a force in the direction of its motion, or in which such a particle would be acted upon by a force tending to displace it, if held *at rest* therein, is said to be a field of electric force. The force which tends to displace a particle of electricity from its normal, or equilibrium position, is called the electric force acting on this particle.

An electric field exists (1) within every substance through which there is a *flow* (or conduction) of electricity, (2) within every dielectric (or insulator) in the vicinity of an electric *charge*, whether this charge be at rest or in motion, and (3) within every dielectric in which there is a *varying* magnetic field.

The electric force exerted on a *positively* charged particle *per unit quantity* of electricity in this particle is called the "intensity" of the electric field, or briefly, the "electric intensity," at this point, and will be represented throughout this book by the symbol F.

Lines drawn in an electric field in such a manner that they coincide in direction at each point with the direction of the electric intensity at that point, *i.e.*, with the direction of the force which would be exerted by the field on a positively charged particle at this point, are called "lines of electric force." Compare with lines of magnetic force.

Since force has the same dimensions as work per unit distance, and since potential difference has the dimensions of work per unit quantity of electricity, the intensity of an electric field

has the same dimensions as potential difference per unit length, or "potential gradient" (see Article 64). The practical unit of electric intensity is therefore the volt per centimeter, or per inch. The c.g.s. electromagnetic unit is the abvolt per centimeter, and the c.g.s. electrostatic unit is the statvolt per centimeter. Note that

$$\begin{aligned} 1 \text{ statvolt per cm.} &= 300 \text{ volts per cm.} \\ &= 3 \times 10^{10} \text{ abvolts per cm.} \end{aligned}$$

Electric intensity, which has the dimensions of *force* per unit charge, should be carefully distinguished from electromotive force, which latter has the dimensions of *work* per unit charge. The relation between the resultant electromotive force e acting around a closed path, or circuit, and the electric intensity F at the various points of this path, is exactly the same as that between

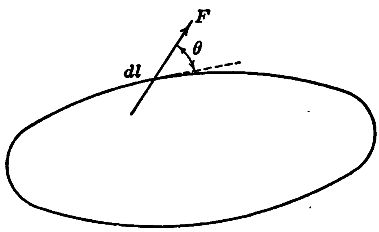


FIG. 99.

mechanical work and mechanical force, viz., the electromotive force acting around the path is equal to the line integral of the electric intensity around this path, or

$$e = \int_o (F \cos \theta) dl \qquad (1)$$

where dl is an elementary length of this path (see Fig. 99), F is the electric intensity at dl, and θ is the angle between the direction of dl and the direction of the electric intensity F, and the integral is taken around the closed path in question.

137. Electric Flux and Dielectric Permeability.—As shown in Article 90, the fundamental relation between the magnetizing force H at each point of a closed loop and the electric currents which thread this loop, is that the line integral of the magnetizing force around the loop is equal to 4π times the algebraic sum of the electric currents which thread it, viz.,

$$4\pi \Sigma\, i = \int_o (H \cos \theta') dl' \qquad (2)$$

In this expression dl' is an elementary length in the loop, H is the magnetizing force at dl', θ' is the angle between the direction of H and the direction of dl', and Σi is the algebraic sum of the currents which thread the loop in the right-handed screw direction, with respect to the direction in which dl' is measured, and all quantities are in c.g.s. electromagnetic units.

Note that $4\pi\Sigma i$ is the magnetomotive force acting around this loop, and therefore equation (2) for the relation between magnetomotive force and magnetizing force H is of exactly the same form as equation (1) for the relation between the electromotive force e acting around a loop and the electric intensity F at each point of this loop.

Experiment shows that when a given loop is threaded by a *varying* electric field, this varying electric field, even though it

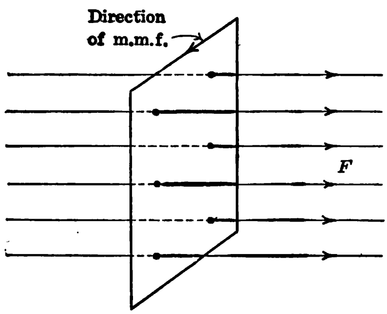


FIG. 100.

produces no flow of electricity through the area enclosed by the loop, always produces a magnetomotive force which acts around this loop and which is proportional to the *time-rate of variation of the electric field.* For example, the loop in question may be in a perfect dielectric, in which case there will be no flow of electricity through the area bounded by the loop. However, if the electric field in this dielectric is *varying with time,* there will be a definite magnetomotive force acting around this loop, *i.e.*, the varying electric field will produce a definite magnetizing force H at each point of the loop.

Experiment shows that the relation between the time-rate of change of an electric field and the magnetomotive force which it produces may be expressed as follows. Consider first a unit area lying in a plane perpendicular to the electric intensity, *i.e.*, perpendicular to the lines of *electric force* through it (see Fig. 100).

Let $4\pi\sigma'$ be the magnetomotive force acting around this unit area, due to the *variation of the electric field which threads this area.* The positive sense of this magnetomotive force is taken in the direction which bears a right-handed screw relation to the lines of electric force. Then

$$4\pi\sigma' = \frac{d}{dt}(kF) \tag{3}$$

where F is the electric intensity at this area and k is a factor whose value depends (1) upon the nature of the medium in which this area is located and (2) upon the system of units employed.

The coefficient k in the above expression may be called the "dielectric permeability" of the medium in which the unit area under consideration is located. The quantity

$$D = kF \tag{4}$$

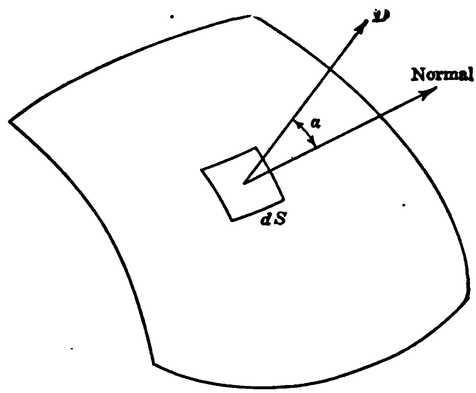


FIG. 101.

is called the "electric flux density" at this area. The direction of the electric flux density is defined as the direction of the electric intensity F. Note that the relation between electric intensity F, electric flux density D, and dielectric permeability k is of exactly the same form as that between magnetizing force H, magnetic flux density B, and magnetic permeability μ.

The surface integral of the electric flux density over any surface in an electric field is called the total "electric flux" through that surface, and may be represented by the symbol ψ, viz.,

$$\psi = \int_S (D \cos \alpha)\, dS \tag{5}$$

where (see Fig. 101) dS is any elementary area in this surface and α is the angle between the direction of the normal to dS and the direction of the electric flux density D at this point.

The electric flux is said to leave that side of the surface from which an outwardly drawn normal makes an angle of less than 90 degrees with the direction of the electric intensity, and to enter the other side of the surface. An equivalent statement of this convention is that the electric flux is said to enter that side of the surface toward which a positive charge would be urged by the electric force, and to leave that side of the surface from which a positive charge would be urged by the electric force.

In the particular case when the surface S is perpendicular at each point to the electric intensity F at this point, *i.e.*, when the surface S is an electric equipotential surface (see Article 64), the relation between the electric flux density D and the electric flux ψ is simply

$$\psi = \int_S Dds \tag{5a}$$

Again, when the electric flux density has the same value at each point of the surface, this becomes

$$\psi = DS \tag{5b}$$

That is, when the electric flux density has the same value at every point of an electric equipotential surface, the electric flux through this surface is equal to the product of the area S of the surface by the electric flux density D at any point in it

Equations (5) to (5*b*) should be compared with the corresponding relations between magnetic flux and magnetic flux density (see Article 87).

From the definitions just given it follows that the *total* magnetomotive force produced in the boundary of any surface S by a varying electric field which threads this surface is $\frac{d\psi}{dt}$. Hence a *varying* electric flux ψ is equivalent, with respect to the magnetic field which it produces (see equations (2) and (3)), to a current of intensity

$$i' = \frac{1}{4\pi}\frac{d\psi}{dt} \tag{6}$$

flowing through the surface under consideration in the same direction as the lines of electric force which intersect this surface.

This equivalent current i', is called the "displacement" current due to the electric flux ψ. The quantity σ' in equation (3) is the *density* of this displacement current.

The relation expressed by equation (6) should be compared with the relation

$$e = -\frac{d\varphi}{dt} \tag{7}$$

between the magnetic flux φ through a surface and the electromotive force e induced in the boundary of this surface.

From equation (6) it is evident that electric flux is proportional to the product of electric current by time, and therefore to quantity of electricity. Hence, the relation between the units of electric flux in the three systems of units is the same as that between the units of quantity of electricity in these three systems of units, viz., 1 c.g.s. electromagnetic unit (abcoulomb) = 10 practical units (coulomb), 1 practical unit (coulomb) = 3×10^9 c.g.s. electrostatic units (statcoulomb).

Problem 1.—Two parallel metallic plates, thoroughly insulated from each other and from all other conductors, are respectively connected by insulated wires to the positive and negative terminals of a battery whose electromotive force is 10,000 volts. A momentary current will flow in the connecting wires, but this flow will cease as soon as the plates become fully charged (see Article 28). The distance between the two plates is 0.5 inch, and the area of each plate is 1 square foot. Except near the edges of the plates the lines of electric force in the dielectric between the plates will be straight and perpendicular to the surfaces of the plates. In the following problem assume all these lines of force to be straight and perpendicular to the plates.

(*a*) What is the electric intensity in the dielectric between the plates in volts per centimeter, abvolts per centimeter, and in statvolts per centimeter? (*b*) If the dielectric between the plates is air, what is the electric flux density at any point between the two plates? The dielectric permeability of air in c.g.s. electrostatic units is 1. (*c*) What is the dielectric permeability of air in c.g.s. electromagnetic units? (*d*) What is the total electric flux through the dielectric between the two plates? (*e*) If this flux increases from zero to its final value in 0.01 second, what is the average displacement current in the dielectric during this interval? (*f*) What is the magnetomotive force acting around the boundary of the dielectric, in a loop parallel to the plates, during this interval? (*g*) If the plates and the dielectric are circular discs (each 1 square foot in area), what is the magnetizing force at each point in the boundary of the dielectric due to this varying electric field? (*h*) What is the magnetizing force at a point in the dielectric at a distance of 5 centimeters from the axis of the disc formed by this dielectric?

Answer.—(*a*) 7870 volts per centimeter = 78.70 × 10^{10} abvolts per centimeter = 26.2 statvolts per centimeter. (*b*) 26.2 c.g.s. electrostatic units per square centimeter = 8.74 × 10^{-10} c.g.s. electromagnetic units per square centimeter. (*c*) ⅙ × 10^{-20} c.g.s. electromagnetic units. (*d*) 24,400 c.g.s. electrostatic units = 0.813 × 10^{-6} c.g.s. electromagnetic units. (*e*) 6.47 × 10^{-6} abamperes = 64.7 microamperes. (*f*) 81.3 × 10^{-6} gilberts = 64.7 × 10^{-6} ampere-turns. (*g*) Circumference of the disc formed by the dielectric is 108 centimeters. Hence, magnetizing force at each point in this circumference is 0.599 × 10^{-6} ampere-turns per centimeter = 0.753 × 10^{-6} gilberts per centimeter. (*h*) The displacement current linked by a circle through the given point concentric with the axis of the disc is 5.46 microamperes, and the magnetizing force at each point of this circle is 0.174 × 10^{-6} ampere-turns per centimeter.

138. Electric Flux and Electric Charge.—Experiment shows that one of the fundamental properties of the electric displacement current is that at any surface in an electric field, *irrespective of the nature of the substances or media on the two sides of this surface,* the rate i_1 at which electricity actually flows up to one side of the surface (by conduction or convection) plus the displacement current i'_1 entering this side of the surface is *always* equal to the rate i_2 at which electricity actually flows away from the other side of this surface plus the displacement current i'_2 leaving this side, viz.,

$$i_1 + i'_1 = i_2 + i'_2 \tag{8}$$

On the other hand, as noted in Article 29, the difference between the rate i_1 at which electricity flows up to a surface and the rate i_2 at which electricity flows away from this surface is always equal to the rate of change of the electric charge on this surface, viz.,

$$i_1 - i_2 = \frac{dq}{dt} \tag{9}$$

where dq is the increase in the amount of positive electricity at this surface in time dt.

Combining equations (8) and (9), it follows that the difference between the displacement current *leaving* one side of the surface and the displacement current *entering* the other side of this surface is likewise equal to the increase in the amount of positive electricity at this surface in time dt, or

$$i'_2 - i'_1 = \frac{dq}{dt} \tag{10}$$

Let ψ_2 be the electric flux which enters one side of this surface and ψ_1 the electric flux which leaves the other side of this surface. Then, from equation (6),

$$\frac{1}{4\pi}\frac{d}{dt}(\psi_1 - \psi_2) = \frac{dq}{dt}$$

Whence,

$$\psi_1 - \psi_2 = 4\pi q \tag{11}$$

where q is the charge on this surface at any instant, and ψ_1 and ψ_2 are respectively the electric fluxes which leave and enter the two sides of this surface at this instant.

Or, considering an electric flux which enters a surface equivalent to a *negative* electric flux leaving this surface, there results the fundamental relation that *the resultant electric flux outward from both sides of a surface is always equal to* 4π *times the algebraic sum of the electric charges which have been conducted* (*or carried by convection*) *to this surface through the media on the two sides of it.* (In this statement a negative charge is considered as a negative quantity.)

From the relation just deduced it follows that at any surface on which there is no electric charge, or at each element of which there exist equal quantities of positive and negative electricity, the electric flux which enters one side of this surface is equal to the electric flux which leaves its other side.

It therefore follows that electric flux may be represented by continuous lines, just as magnetic flux is represented by continuous lines, and that these lines must either (1) form closed loops or (2) originate from a positively charged surface and end on a negatively charged surface. When used in a quantitative sense, the term lines of electric force is to be understood to mean lines drawn in the field in such a manner that their direction at each point coincides with the direction of the electric flux density at this point, and their number entering or leaving any surface is equal to the electric flux which enters or leaves this surface. Such lines are also called lines of electric induction.[1]

[1] Some physicists make the same distinction between lines of electric force and lines of electric induction as is sometimes made between lines of magnetic force and lines of magnetic induction (see footnote, p. 243, Part I.). Throughout this book, however, the terms "lines of electric force" and "lines of electric induction" are used synonymously.

From equation (11) it follows that 4π lines of electric force originate at every unit positive charge and end on a unit negative charge. Or, every line of electric force which has ends must have a positive charge of $\frac{1}{4\pi}$ units at one end and a negative charge of $\frac{1}{4\pi}$ units at the other end.

139. Specific Inductive Capacity or Dielectric Constant.— From the definition of electric flux density (see equation (4)) it follows that the electric flux density D at any point depends (1) upon the electric intensity F at this point and (2) upon the nature of the medium at this point. Consequently, the same electric intensity F will in general produce in different media different values of the electric flux density D. For example, calling k_s the dielectric permeability of free space, or vacuum, and k the dielectric permeability of any substance (*e.g.*, air, glass, or other insulator), then for the same electric intensity F in the two media, the ratio of the electric flux densities in the two media is

$$\frac{D}{D_s} = \frac{k}{k_s} = K \tag{12}$$

This ratio K, which may be defined as the ratio of the electric flux density in the given substance, corresponding to a given electric intensity, to the electric flux density which this same electric intensity would produce in free space, is called the "specific inductive capacity," or "dielectric constant," of the given substance. The term dielectric constant, though frequently employed, is somewhat of a misnomer, for the ratio K is not always a constant.

It is important to note that the *specific inductive capacity* of a substance is simply a numerical ratio, and is independent of the system of units employed, provided the two flux densities D and D_s are expressed in the same unit. The dielectric *permeability* k, however, is not a numerical ratio, but has the dimensions of electric charge per unit area divided by force per unit charge. Its numerical value therefore does depend upon the system of units employed.

The c.g.s. electrostatic system of units is based on the arbitrary choice of unity as the numerical value of the dielectric permeability of free space. In this system of units, therefore, the dielectric permeability of any medium is numerically equal to its spe-

cific inductive capacity. In the c.g.s. electromagnetic system, on the other hand, the dielectric permeability k is equal to $\frac{1}{9 \times 10^{20}}$ times the specific inductive capacity K of the given medium, or

$$k \text{ in c.g.s. electromagnetic units} = \frac{K}{9 \times 10^{20}}$$

Similarly, in the practical system of units, when the centimeter is used as the unit of length, the dielectric permeability k is equal to $\frac{1}{9 \times 10^{11}}$ times the specific inductive capacity K, or

$$k \text{ in practical units} = \frac{K}{9 \times 10^{11}}$$

From the definition above given, the specific inductive capacity of free space is equal to unity, irrespective of the system of units employed. In the c.g.s. electrostatic system of units, the dielectric permeability of free space is likewise equal to unity. In the c.g.s. electromagnetic system of units, however, the dielectric permeability of free space is equal to $\frac{1}{9 \times 10^{20}}$. Throughout the remainder of this chapter, unless distinctly stated otherwise, where the specific inductive capacity (*capital K*) is used in a formula, the formula will be given in c.g.s. electrostatic units. Any given data may be readily reduced to this system of units by employing the proper conversion factors. See any electrical engineers' handbook for tables of conversion factors.

The specific inductive capacity of a substance may be readily measured by a number of methods. The principle involved in these methods of measurement is discussed in Article 154. Such measurements show that the specific inductive capacity of air and other gases is substantially equal to that of free space, viz., unity. Liquid and solid insulators (such as oil, glass, porcelain, etc.) have a specific inductive capacity ranging in value from about 2 to 10, depending upon the particular substance in question.

Substances which are neither good insulators nor good conductors, such as water, have relatively high specific inductive capacity; *e.g.*, the specific inductive capacity of water is about 80. Good conductors, such as metals, probably have very large specific inductive capacity, although very little is known about the actual values of the specific inductive capacity of such

substances, since the conduction currents in such substances completely mask the effect, if any, of any displacement current, which may be established in them.

The specific inductive capacity of solid and liquid dielectrics is not strictly a constant for any particular substance, but depends, among other things, upon the electric intensity established in it, in much the same manner as the magnetic permeability of ferromagnetic substances depends upon the magnetizing force. This variation in the value of the specific inductive capacity, however, is much less pronounced than the variation of the magnetic permeability of ferromagnetic substances.

140. Electric Fields In and Around Charged Conductors.—As shown in Article 64, the electric intensity at any point in a substance through which there is a flow of electricity or conduction current, is always equal to the product of the resistivity ρ of this substance by the density σ of the conduction current at this point, viz.,

$$F = \rho\sigma \tag{13}$$

That is, the electric intensity in a substance is always equal to the *resistance drop* through it *per unit length*. Consequently, *within the substance of a conductor through which there is no flow of electricity there can be no electric field*, *i.e.*, in a conductor in which there is no "conduction" current the electric intensity F is always zero.

This relation is true, whether the substance in question is a good conductor or a poor conductor. Hence, since every insulator is a conductor, to at least a slight extent, an electric field cannot be established in any substance without causing at least a small conduction current to flow through it. However, in the case of good insulators, such as glass, porcelain, rubber, and the various insulating compounds used in practice, the resistivity ρ is so large that the conduction current is usually practically negligible.

The electric flux density D at any point in a substance is, by definition,

$$D = kF \tag{14}$$

where F is the electric intensity at this point and k is the dielectric permeability of the substance. Consequently, since *in a conductor* in which there is no flow of electricity, there can

be no electric intensity, it follows that in such a conductor the electric flux density is zero. Hence, *in a conductor through which there is no flow of electricity there can be no electric flux, or lines of electric force.*

Consequently, when two or more charged conductors are insulated from one another by a perfect insulator, and the charges on these conductors are at rest, the only place in which an electric field can exist is in the medium which surrounds the conductors. The lines of electric force, being confined solely to the insulating medium, must therefore end on the conductors, each line originating from a positive charge of $\frac{1}{4\pi}$ units and ending on an equal negative charge.

When an electric field is established in a dielectric which is not a perfect insulator, there will be in general a flow of electricity both through it and the conductors which it separates, and consequently an electric field may also exist within these conductors. However, when the conductors are metals or other substance of relatively low resistivity, the electric intensity within the conductors is generally inappreciable in comparison with the electric intensity in the surrounding dielectric. Therefore in this case also the electric flux is confined practically to the dielectric.

Consider any point in any surface whatever in an electric field, and imagine a rectangular closed path drawn around this point in such a manner that the long sides of this rectangle are on opposite sides of this surface. By applying to this closed loop the general relation expressed by equation (1), it may be readily shown that the tangental components of the electric intensity on the two sides of a surface are always equal. (Compare with the corresponding relation for the tangental components of the magnetizing force, Article 94.)

As a consequence of this relation, and the fact that in conductors in which there is no flow of electricity the electric intensity is always zero, it follows that the tangental component of the electric intensity in the dielectric just outside the surface of a conductor in which there is no electric current, is always zero. Hence *the lines of electric force always leave and enter the surface of a conductor at right angles to this surface,* provided there is no current in the conductor.

Consider a unit area in the surface of a charged conductor

and let σ be the charge on this area, *i.e.*, the density of the charge on this surface. This area will be the origin or terminus of $4\pi\sigma$ lines of force, accordingly as the charge on this surface is positive or negative. Hence in the dielectric just outside this area the electric flux density is

$$D = 4\pi\sigma \tag{15}$$

and the electric intensity is

$$F = \frac{4\pi\sigma}{k} \tag{15a}$$

where k is the dielectric permeability of this dielectric. The direction of this electric intensity is always *perpendicular* to the surface, *outward* from the surface when the charge on it is positive and toward the surface when the charge on it is negative.

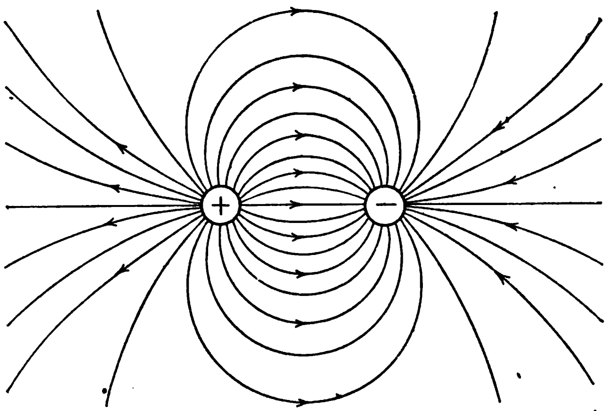


FIG. 102.—Lines of electric force due to two equally and oppositely charged spheres.

These relations, though strictly true only when there is no current in the conductor and no magnetically induced electromotive force in the dielectric, are also practically true for a conductor in which there *is* an electric current, constant or varying, provided the resistivity of the conductor is small in comparison with the resistivity of the surrounding dielectric, and provided the rate of change of the current is not exceedingly rapid.

To sum up, a positive charge of q units on the surface of a good conductor is always the origin of $4\pi q$ lines of electric force, which, in the dielectric just outside the conductor, are at right angles to this surface. Similarly, a charge of q units of negative

electricity on the surface of a good conductor is always the end of $4\pi q$ lines of force, which enter this surface at right angles. This relation between the direction of the lines of force and the surface of a conductor applies only to points infinitely close to the conductor. In the space between the two conducting surfaces the lines of electric force are curves, as indicated in Fig. 102 for the particular case of two spheres charged with equal amounts of positive and negative electricity respectively.

141. Difference of Electric Potential and Electric Intensity.—From the definitions and experimental facts stated in Chapter II, it follows that when there is no source of electromotive force in the path connecting two points (*i.e.*, when there is no contact electromotive force or magnetically induced electromotive force in this path), the difference of electric potential between these

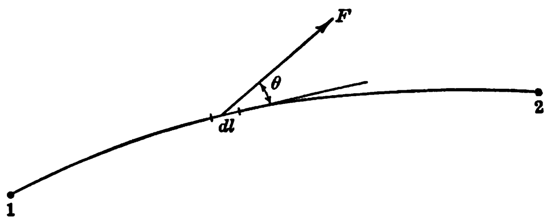


FIG. 103.

two points is equal to the resistance drop along this path from one point to the other. As noted in Article 64, the resistance drop per unit length at any point is equal to the electric intensity at this point, viz., to the electric force per unit charge. Hence *the drop of electric potential from any point* 1 *to any other point* 2 *is equal to the line integral of the electric intensity from* 1 *to* 2 *along any path in which there is no contact electromotive force and no magnetically induced electromotive force.* This relation is a perfectly general one, and is applicable to both insulators and conductors.

The relation just stated may be conveniently expressed mathematically as follows: Let v_{12} designate the drop of electric potential from the point 1 to the point 2, and let dl be an elementary length in any path from 1 to 2 in which there is no contact or magnetically induced electromotive force, let F be the electric intensity at dl, and let θ be the angle between the direction of dl and the direction of F. Then the drop of electric potential from 1 to 2 is

$$v_{12} = \int_1^2 (F \cos \theta) dl \tag{16}$$

From equation (13) the electric intensity F may be expressed in terms of the density σ of the conduction current at dl and the resistivity ρ of the medium at dl. From equation (14) F may also be expressed in terms of the electric flux density D at dl and the dielectric permeability k of the medium at this point. Equation (16) may therefore also be written

$$v_{12} = \int_1^2 \rho (\sigma \cos \theta) dl \tag{16a}$$

or

$$v_{12} = \int_1^2 \frac{1}{k} (D \cos \theta) dl \tag{16b}$$

Equation (16*a*) is applicable only when there is an actual flow of electricity, and equation (16*b*) is applicable only when an electric flux exists in the medium under consideration. In the case of a substance which has both a finite resistivity ρ and a finite dielectric permeability k, as, for example, a "leaky" insulator, either equation may be employed.

Equation (16*b*) shows that whenever there exists a difference of electric potential between two conductors separated by a dielectric (for example, the two wires of a transmission line), there must be an electric field in the medium surrounding these conductors, and that the lines of force which represent this electric field must originate at and end on the two conductors. Consequently, these conductors must be equally and oppositely charged. In short, *whenever a difference of electric potential exists between two or more conductors, the surfaces of these conductors are always charged with electricity, and* (from article 140) *the sum of all the positive charges in the field must be equal to the sum of all the negative charges in the field.*

142. Electric Flux Density at any Point Due to Unvarying (Static) Electric Charges.—Imagine a sphere of conducting material to be charged with q units of positive electricity (see Fig. 104), and let this sphere be at an infinite distance from all other conductors, and let it be surrounded by a uniform medium whose dielectric permeability is k. From symmtery, the lines of electric force due to the charge q must then be uniformly distributed radial lines extending out into space in all directions.

When there is no flow of electricity within the sphere, *i.e.*,

when the electricity on it is at rest, there will be no electric field within it. Therefore, from the relation expressed by equation (11), there will be $4\pi q$ lines of electric force proceeding outward from the sphere, and these lines will intersect at right angles every spherical surface in the dielectric concentric with the given sphere. Let x be the radius of such a surface. Then the area of this surface is $4\pi x^2$. The number of lines of electric force through unit area of this surface is then

$$\frac{4\pi q}{4\pi x^2} = \frac{q}{x^2}$$

Hence the electric flux density at any point P outside of, and

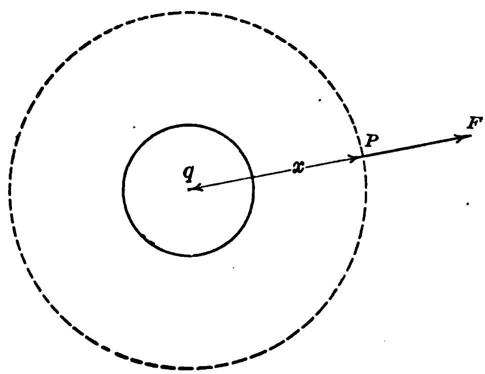


FIG. 104.

at a distance x from the center of, a charged conducting sphere on which the charge is at rest is

$$D = \frac{q}{x^2} \tag{17}$$

where q is the charge on this sphere. Note that this flux density is *independent* of the nature of the medium, *provided* the medium is *uniform, i.e.*, has the same dielectric permeability at every point. Note also that this flux density is independent of the radius of the sphere on which the charge is located, but depends only upon the total charge on the sphere and the distance of the point P from its center.

From equation (4) and equation (17) it follows that the electric intensity at the point P is

$$F = \frac{q}{kx^2} \tag{18}$$

where k is the dielectric permeability of the medium. The electric intensity therefore does depend upon the nature of the medium surrounding the sphere, whereas, when the medium is uniform, the electric flux density does not.

When the charge q is positive, both the flux density and the electric intensity are in the direction of the line drawn from the center of the sphere to the point P. When q is negative, then the values of D and F given by these equations are likewise negative, indicating that the flux density and electric intensity are in the opposite direction, *i.e.*, from the point P toward the center of the sphere.

Equations (17) and (18) are applicable only to the special case of a single conducting sphere surrounded by a uniform medium.

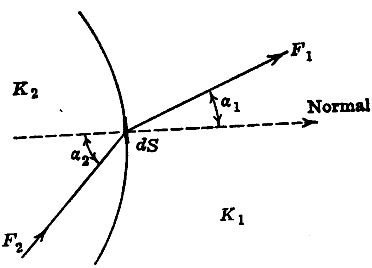


FIG. 105.

In the general case of any number of conductors of any shape, surrounded by any number of dielectrics, experiment shows that the resultant electric intensity and electric flux density at any point due to any number of *unvarying* electric charges depend upon the value and distribution of these charges in exactly the same way that the resultant magnetizing force and magnetic flux at any point in a magnetic field depend upon the value and distribution of these magnetic poles (see Article 103).

Consider any elementary area dS in the surface of contact between any two different substances in the field (Fig. 105), and let $F_1 \cos \alpha_1$ be the normal component of the electric intensity on that side of this surface at which the lines of force leave and let $F_2 \cos \alpha_2$ be the normal component of the electric intensity on the other side of this surface. Put

$$dq' = \frac{k_s}{4\pi}(F_1 \cos \alpha_1 - F_2 \cos \alpha_2)\, dS$$

2

where k_s is the dielectric permeability of free space in the particular system of units employed. The quantity dq' as defined by this equation is called the "total equivalent" charge at the surface dS.

This total equivalent charge also may be expressed in terms of the electric flux densities D_1 and D_2 on the two sides of the surface, and the dielectric permeabilities k_1 and k_2 of the substances which are in contact at this surface, viz.,

$$dq' = \frac{k_s}{4\pi}\left(\frac{D_1 \cos\alpha_1\, dS}{k_1} - \frac{D_2 \cos\alpha_2\, dS}{k_2}\right)$$

But from equation (5), $D_1 \cos \alpha_1\, dS$ is equal to the number of lines of force, say $d\psi_1$, which leave dS, and $D_2 \cos \alpha_2 dS$ is equal to the number of lines of electric force, say $d\psi_2$, which enter dS. Also note that $\frac{k_1}{k_s} = K_1$ and $\frac{k_2}{k_s} = K_2$ are the specific inductive capacities of the media on the two sides of this surface. Hence the above expression for the total equivalent charge at the surface dS may be written

$$dq' = \frac{1}{4\pi}\left(\frac{1}{K_1}\, d\psi_1 - \frac{1}{K_2}\, d\psi_2\right) \qquad (19)$$

On the other hand, from equation (9), the difference between $d\psi_1$ and $d\psi_2$ is equal to 4π times the charge which this surface has acquired as the result of the *transfer of electricity to it through one or the other of the media on the two sides of this surface.*

Let dq be the value of the charge thus transferred to this surface. This charge dq will be referred to as the "real" charge on this surface. Then $d\psi_1 - d\psi_2 = 4\pi dq$, or

$$d\psi_2 = d\psi_1 - 4\pi dq$$

Equation (19) may then be written

$$dq' = \frac{1}{K_2} dq + \frac{d\psi_1}{4\pi}\left(\frac{1}{K_1} - \frac{1}{K_2}\right) \qquad (19a)$$

Hence the total equivalent charge at any surface of finite area S is

$$q' = \frac{q}{K_2} + \frac{\psi_1}{4\pi}\left(\frac{1}{K_1} - \frac{1}{K_2}\right) \qquad (19b)$$

where q is the total real charge on this surface, ψ_1 is the total number of lines of electric force which leave this surface, K_1 is the specific inductive capacity of the medium on that side

of the surface from which the lines of force leave, and K_2 is the specific inductive capacity of the medium on the other side of this surface.

Experiment shows that when the total equivalent charge at every surface, as defined by equation (19*b*), is taken into account, and these charges remain constant in magnitude and distribution, the resultant electric intensity at any point P in the field is equal to the vector sum of the electric intensities at P as calculated from the formula

$$dF = \frac{dq'}{k_s x^2} \tag{20}$$

for each of these equivalent charges. That is, the resultant electric intensity at each point in an unvarying electric field is

$$F = \frac{1}{k_s} \overline{\sum} \frac{dq'}{x^2} \tag{20a}$$

where the symbol $\overline{\Sigma}$ indicates a *vector* summation. In this formula k_s is the dielectric permeability of free space (= unity in the c.g.s. electrostatic system), and x is the distance of the point P from any particular equivalent charge dq'. Note particularly that k_s is *not* the dielectric permeability of the medium at P, unless this happens to be free space.

Problem 2.—A circular metal plate has a real charge of σ statcoulombs per square centimeter on one face only, the opposite face being uncharged. The radius of the disc is r centimeters. The dielectric in contact with this disc has a specific inductive capacity K.

(*a*) Prove that the total equivalent charge *per unit area* of the charged surface of the disc is

$$\sigma' = \frac{\sigma}{K}$$

(*b*) Prove that at any point on the axis of the disc, at a distance a from its charged surface, whether this point be in the dielectric, in the substance of the disc, or on the opposite side of the disc, the electric intensity due to the equivalent charge at the charged surface of the disc is, in statvolts per centimeter,

$$F_1 = \frac{2\pi\sigma}{K}(1 - \cos\theta) \tag{21}$$

where

$$\theta = \tan^{-1}\left(\frac{r}{a}\right)$$

(See Problem 6 of Article 134.)

(*c*) Show that when a is small in comparison with r, the electric intensity due to the equivalent charge at the charged surface of the disc is

$$F_1 = \frac{2\pi\sigma}{K} \tag{21a}$$

(*d*) Consider a second metalilc disc of radius r placed co-axially with and parallel to the first disc, and let the face of this disc opposite the charged face of the first disc be charged with $-\sigma$ statcoulombs (real charge) per square centimeter, its opposite face being uncharged. Show that at any point on the axis of the two discs, *in the dielectric between their charged surfaces*, the total electric intensity, in statvolts per centimeter, is

$$F = \frac{4\pi\sigma}{K} \tag{21b}$$

provided the distance apart of the discs is small compared with their radius.

(*e*) For the same conditions as specified in (*d*) show that at any point on the axis of the two discs, *either within the substance of the disc or in the dielectric outside the space between the two*, the electric intenstiy is practically zero, provided the point in question is at a distance from each charged surface small in comparison with the radius of this surface.

143. Real and Apparent Charges.—In equations (19*a*) to (20*a*) the symbol q is used to designate the charge which is *conducted* to a surface through the substance on one side or the other of the surface, or which is brought to this surface by convection. As already noted, this charge may be called the "real" charge at the given surface. The *algebraic* difference between the total *equivalent* charge q' and this conducted charge q, viz.,

$$q_a = q' - q \tag{22}$$

will be referred to as the "apparent" charge on the given surface.

From equation (19*b*) an apparent charge exists at the surface of contact between every two media whose specific inductive capacities are different, except in the special case of the surface of a good conductor (practically infinite dielectric constant) in contact with a medium whose specific inductive capacity is unity.

When there is no real charge at this surface of contact ($q = 0$) the apparent charge is

$$q_a = \frac{\psi}{4\pi}\left(\frac{1}{K_1} - \frac{1}{K_2}\right) \tag{22a}$$

where ψ is the electric flux through this surface, K_1 is the specific inductive capacity of the medium on that side of the surface from which the flux leaves, and K_2 is the specific inductive capacity of the medium on the other side of this surface.

For example, consider a dielectric, whose specific inductive capacity is K, to be placed in an electric field in air (specific inductive capacity unity). On that portion ofthe surface of this

dielectric from which the lines of electric force *leave* there will be an apparent charge equal to

$$\frac{\psi}{4\pi}\left(1 - \frac{1}{K}\right) \tag{22b}$$

and on that portion of the surface of this dielectric at which the lines of electric force *enter* there will be an apparent charge equal to

$$\frac{\psi}{4\pi}\left(\frac{1}{K} - 1\right) = -\frac{\psi}{4\pi}\left(1 - \frac{1}{K}\right) \tag{22c}$$

The surface of a dielectric in an electric field may therefore always be divided into two portions which are the seats of equal and opposite apparent charges. Compare with the magnetic poles induced at the surface of a magnetic substance in a magnetic field, Article 103.

The apparent charge at the surface of contact between a metallic conductor and a dielectric whose specific inductive capacity is K

$$q_a = q\left(\frac{1}{K} - 1\right) = -q\left(1 - \frac{1}{K}\right) \tag{22d}$$

where q is the real charge at this surface. This follows directly from equations (22) and (19*b*), noting that the dielectric constant of a metallic conductor is practically infinite and that the flux which leaves a real charge q on a conductor is equal to $4\pi q$. Since K is never less than unity, the apparent charge at such a surface is always of *opposite* sign to the real charge at this surface.

Problem 3.—A piece of plate glass is placed in a uniform electric field in air, with its flat surface perpendicular to the lines of electric force which represent this field. The electric intensity in the air just outside the glass is 7500 volts per centimeter. The dielectric constant of the glass is 7. The plate has a thickness of 0.5 centimeter, and each flat surface an area of 100 square centimeters.

(*a*) What is the total electric flux which enters the glass? (*b*) What is the total electric flux which passes through the glass? (*c*) What is the electric flux density inside the glass? (*d*) What is the electric intensity inside the glass? (*e*) What is difference of potential between the two faces of the glass plate? (*f*) What is the apparent charge on each face of the glass? (*g*) Were one face of the glass in contact with a metal plate, the electric flux leaving the other face of the glass remaining the same as in (*b*), what would be the real charge on the metal plate? (*h*) What would be the apparent charge on the glass? (*i*) What would be the total equivalent charge at the surface of contact between the metal and glass plates?

Answer.—(*a*) 2500 c.g.s.e.s. units. (*b*) 2500 c.g.s.e.s. units. (*c*) 25 c.g.s.e.s. units. (*d*) 3.57 statvolts per cm. = 1071 volts per cm. (*e*) 536 volts. (*f*) −171 statcoulombs on the face at which the lines of electric force enter and +171 statcoulombs on the face from which the lines of force leave the glass. (*g*) +199 statcoulombs. (*h*) −171 statcoulombs. (*i*) +28 statcoulombs.

144. Hypothesis Regarding the Nature of the Apparent Charges at the Surface of a Dielectric.—On the basis of the electron theory, the apparent charge at the surface of a dielectric is actually a real charge of electricity, differing from a charge which is *conducted* to a surface only in that it cannot be removed from the molecules which make up this surface. This conception arises from the assumption that the fundamental difference between a conductor and a dielectric is that a conductor contains

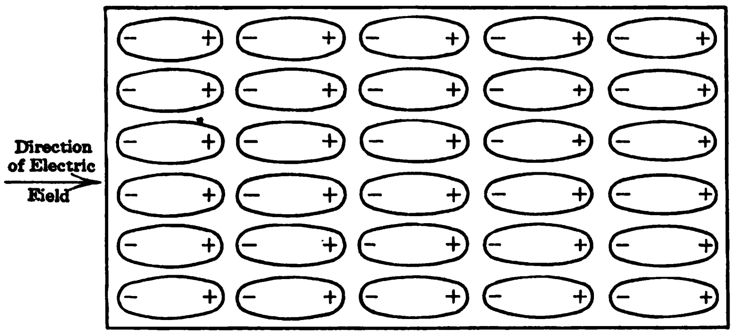


Fig. 106.

a large number of "free" electrons which can move from one portion of the conductor to another, and may even be completely removed from the conductor, without destroying its nature, whereas in a dielectric most of the electrons present are each permanently attached to, and form a part of, a definite molecule, and cannot be removed from it without destroying it.

When an electric field is established in a conductor, the free electrons within it are acted upon by a force in the opposite direction to the electric intensity (for the electrons are negative charges), and their motion through the conductor is opposed only by a force of a frictional nature. However, when an electric field is established in a dielectric, the electron which forms a permanent part of a given molecule cannot move out of this molecule, but is displaced a minute distance with respect to the positive neucleus of the molecule, and the molecule as a whole turns so that a line

drawn through it from its negatively charged end to its positively charged end coincides with the direction of the electric field.

Such a displacement of the electrons within the molecules of a dielectric, and the turning of these molecules, will result in a volume distribution of charges somewhat like that shown in Fig. 106. By applying the formula $F = \frac{q}{k_e x^2}$ to each of these molecular charges, it may be shown that such a volume distribution of charges will produce identically the same electric field as that which would be produced by a surface distribution of charge satisfying the relations given in Article 103.

This theory of the nature of a dielectric should be compared with the theory of the molecular nature of ferromagnetic substances (Article 114). Note however, that in a dielectric it is the particles of electricity which are arranged along definite lines by an electric force, whereas in a magnet it is the axes of the circuits (or orbits) in which these particles move which are so arranged. A dielectric in an electric field is frequently said to be electrically "polarized," just as a piece of iron in a magnetic field is said to be magnetically "polarized."

145. Distribution of Electric Charges on Conductors and Dielectrics.—When the strength and distribution of all the electric charges, both real and apparent, in a given field are known, the electric intensity at each point in the field due to these charges may always be calculated[1] from equation (20*a*) of Article 142. Since the distribution of the charges depends in turn upon the electric intensity at each surface of contact between dissimilar substances, the exact determination of the value of the electric intensity, except in certain special cases of symmetry, is by no means a simple problem.

From the relations above developed, however, certain general conclusions of both practical and theoretical importance may be drawn, viz.:

1. As noted in Article 141, the electric intensity within the substance of a conductor through which there is no flow of electricity is always zero. Hence one condition which must be satisfied by the equivalent charges in an electric field is that they

[1] This equation gives the total electric intensity only when the magnetic field in the region under consideration is not varying, *i.e.*, when there is no appreciable magnetically induced electromotive force in this region.

must always be **so distributed that they produce zero electric intensity within the substance of every conductor in the field,** provided there is no flow of electricity through this conductor, and also provided there is no magnetically induced electromotive force within this conductor. This law refers to the combined effect of the real and apparent charges.

2. Experiment justifies the assumption that at any elementary surface (large in comparison with the size of a molecule) *inside* a uniform substance, the resultant real charge and the resultant apparent charge are both zero. This is true whether or not there is a flow of electricity through the substance. Hence **both the real and the apparent charges on a body are always confined to its surface.**

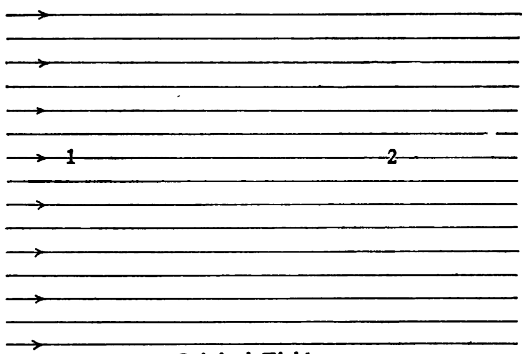
3. Within the substance of a dielectric, the electric intensity is not zero, but is equal to the electric flux density at this point divided by the dielectric permeability at this point. The equivalent charges in the field must therefore also be distributed in such a manner that equation (20a) is satisfied at every point in the field, and in addition, in such a manner that equation (19a) is satisfied at every point of every surface of contact between two different substances or media.

4. As a consequence of the fact that the resultant electric intensity inside a conductor in which there is no current and no magnetically induced electromotive force is always zero, it may be shown that the real charges on the surface of a conductor tend to distribute themselves in such a manner that the density of the charge is greatest where the curvature of the surface is greatest. A sharp point in the surface of a conductor is therefore usually the seat of a relatively dense charge, and the electric intensity in the dielectric just opposite such a point is relatively high (see equation (15a)). As the charge on a conductor is increased, the surrounding dielectric will therefore as a rule break down first where it is in contact with that portion of the conducting surface which has the greatest curvature.

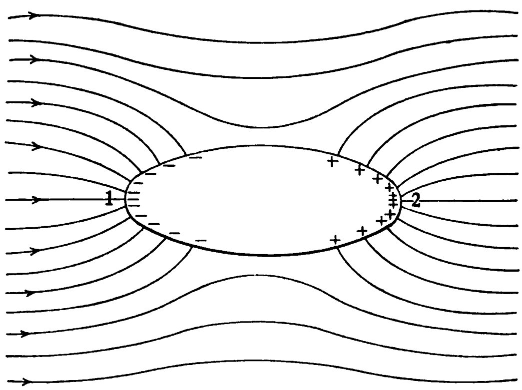
146. Electrostatic Induction.—Consider an electric field, such as that represented by the lines of force shown in the upper part of Fig. 107, and let there be no contact electromotive force or magnetically induced electromotive force in the region under consideration. Any surface in such a field which is perpendicular at each point to the line of electric force through that point is an electric

equipotential surface, for the line integral of the electric force along any path from one point to another in such a surface is zero (see Article 64).

Since in a conductor in which there is no electromotive force and in which there is no flow of electricity, there can exist no electric field, it follows that the surface of every conductor in an



Original Field



Field after Insertion of Conductor

FIG. 107.

electric field is an electric equipotential surface, provided there is no electromotive force in it and no flow of electricity through it. Hence, when a conductor is placed in an electric field, such as shown in the upper part of Fig. 107, this conductor must become an electric equipotential surface, when there is no current in it. That is, when the points 1 and 2 in the upper part of Fig. 107, between which a difference of potential exists, are connected by

a completely insulated conductor, these two points must come to the same potential.

Experiment shows that this condition results in the establishment of equal and opposite charges on the surface of the conductor, distributed as indicated in the lower part of Fig. 107. The charges thus established on a conductor are called "induced" charges, and the entire phenomenon is referred to as "electrostatic induction." These induced charges change the shape and distribution of the electric lines of force as indicated in the figure.

When the given conductor is completely insulated from all other conductors, the only source of electricity from which these charges can come is the conductor itself. Hence, during the establishment of these induced charges, there must be a flow of electricity through the conductor in the direction of the original field, *i.e.*, from 1 to 2, but this flow ceases as soon as the new distribution of charges is such as to produce zero electric intensity within the conductor. On the basis of the electron theory, this flow of electricity is looked upon as being an actual motion of electrons (negative electricity) in the opposite direction to that of the electric intensity, *i.e.*, from 2 to 1, leaving the surface of the conductor in the vicinity of 2 positively charged, and producing a negative charge on the surface of the conductor in the vicinity of 1.

That there is an actual transfer of electricity from one part of the conductor to the other is borne out by the experimental fact that when a conductor is thus charged by electrostatic induction, the two halves of the conductor may be separated, and one portion will remain positively charged and the other portion negatively charged, even when removed completely from the field. The distribution of the charges on the two portions, however, will in general change when the two parts are separated.

When a dielectric is placed in an electric field, a similar phenomenon is produced, with this important difference, viz., there is no actual transfer of electricity from one portion of the dielectric to the other (assuming the dielectric to be a perfect insulator), although *apparent* charges do appear at the two end surfaces of the dielectric. The effect, when the dielectric has a greater specific inductive capacity than that of the surrounding medium, is entirely analogous to the effect produced in a piece of iron

when placed in a magnetic field in air (see Fig. 69), *i.e.*, the dielectric becomes electrically polarized as explained in Article 144.

When a conductor is placed in an electric field, the charges induced on its surface completely neutralize the original field within the space which it is made to occupy. The lines of electric force which enter it terminate on that portion of its surface which is negatively charged, and an equal number of lines of electric force originate at the positive charge induced on the rest of its surface. *No lines of electric force, however, pass through the conductor* (see Fig. 107).

When a dielectric is placed in an electric field in a medium whose specific inductive capacity is *less* than that of this dielectric, the charges induced on its surface do not completely neutralize the original electric intensity in the space which it is made to occupy, but merely *reduce* the resultant *electric intensity* in this space. The electric *flux density* in this space, however, is *increased.* That is, the lines of electric force crowd into the dielectric, just as the lines of magnetic force crowd into a piece of iron (compare Fig. 69).

When the surrounding medium has a *greater* specific inductive capacity than that of the dielectric which is inserted in the field, a positive apparent charge is induced on the end of the dielectric at which the lines of force enter, instead of on the end at which they leave (see Article 143). An equal negative apparent charge is induced on the other end of the dielectric. In this case, therefore, the resultant *electric intensity* in the space which the dielectric is made to occupy is *greater* than that due to the original field. The electric *flux density*, however, is *less* than that originally in this space, *i.e.*, the presence of the dielectric makes the lines of force crowd into the surrounding medium. An entirely analogous effect is produced in a magnetic field in air when a *diamagnetic* substance is placed in this field.

147. Induced Charges on Conductors.—A fact of fundamental importance, and which follows as a necessary consequence of the general principles already set forth, is that when any number of charged conductors, on which the charges are at rest, are surrounded by a closed conducting shell, completely insulated from all other conductors (including the earth), **there is induced on the outside surface of this shell a charge equal to the algebraic sum of all the charges on these conductors, and on the inside**

surface of this shell an equal and opposite charge. This follows immediately from the fact (1) that no lines of force can pass through the substance of a conductor in which there is no flow of electricity, and the fact (2) that the charges induced on an insulated conductor are always equal an opposite.

When an insulated conductor on which charges are induced is connected to another conductor, the distribution of charges will of course be altered. The fundamental relation is always that the distribution of the charges must be such as to produce zero electric intensity in *all* the conductors in the field. This usually means that the positive and negative charges induced on a conductor, or on a system of conductors all of which are connected to one another, tend to get as far apart as possible. Hence, when a conductor in an electric field is connected to the earth, the charge which is repelled by the field will be repelled to such a distance that its effect may usually be neglected, *i.e.*, it may be looked upon as being repelled to the "other side of the earth."

For example, when a closed shell surrounding one or more charged conductors is connected to the earth, the charge induced on the outside of the shell flows off to the earth, and no appreciable electric field exists in the vicinity of the shell. This applies only to the region outside the shell. Inside the shell there is a field due to the charged conductors within it and the charge induced on its inside surface. Practical application is made of these relations in calculating the electrostatic capacity of cables (see Article 150).

Another important fact, which follows directly from the principle that no difference of electric potential can exist in a conductor in which there is no flow of electricity and no electromotive force, is that a **region completely surrounded by a closed conducting shell is completely screened from all external electrostatics effects.** By an electrostatic effect is meant the effect due to electric charges at rest.

Such a screen, if a good conductor, is also a protection against effects due to electricity in motion, *i.e.*, against electric forces due to varying magnetic fields. This is due to the fact that any tendency to establish a difference of potential between points inside such a screen immediately sets up a flow of electricity in it tending to prevent this rise of potential. In fact, one or two wires

connected to the earth are often sufficient to maintain a safe potential gradient in their vicinity, even when the inducing field is extremely intense, as, for example, in the case of lightning. This is the principle of the lightning rod and of the "ground" wires commonly used over, or along side of, high-voltage transmission lines.

148. Dielectric Strength, Electric Spark and Electric Corona.—Experiment shows that when an electric field is established in a dielectric and the intensity of this field is increased, a point is reached at which the dielectric loses its insulating property and becomes a relatively good conductor. This condition is usually manifested by a spark, which burns a hole through the dielectric, *i.e.*, the dielectric is "punctured."

Under other conditions the breakdown may not be permanent, but may result in the acquisition of a high conductivity by the dielectric only while the voltage gradient is maintained above the critical value, the dielectric regaining its insulating property when the field is reduced below this critical value. This latter condition is usually described as the formation of an electric "corona" in the dielectric. In the case of air or other gases the formation of corona manifests itself by a bluish light in the dielectric around the conductors between which the field is established. Whether the breakdown produced by a given voltage is of the nature of a puncture, or results in the formation of a corona, depends chiefly upon the *distribution* of the dielectric flux produced in the dielectric. See the article on *Corona, Electric,* in Pender's *Handbook for Electrical Engineers.*

The critical field intensity, or voltage gradient, at which breakdown occurs is called the "dielectric strength" of the dielectric. The dielectric strength depends upon the nature of the dielectric its value for the various dielectrics ordinarily employed in practice depending decidedly upon their chemical and physical nature. It is also found that for a given dielectric the critical voltage *gradient* at which breakdown occurs depends in general upon (*a*) the distribution of the dielectric flux just prior to breakdown and (*b*) upon the thickness of the dielectric. It is naturally to be expected that the *total voltage* (total potential difference) required to produce a breakdown would depend upon the distribution of the dielectric flux and the thickness of the dielectric, for the voltage gradient at all voltages depends upon these factors

(see Article 150), but there is as yet no satisfactory explanation of the dependence of the *critical gradient* upon these factors.

In fact, but little is known regarding the nature of an electric breakdown, and even the values of the dielectric strength are known only approximately in most instances, for in many of the tests made to determine its value the distribution of the electric flux was not known. The values of the dielectric strength given in various text-books and handbooks must, therefore, be considered only as rough approximations, except for conditions identical with those under which the tests were made.

149. Absolute Electric Potential Due to Unvarying Electric Charges.—As noted in Article 141, the drop of electric potential between any point 1 and any other point 2 in an electric field is equal to the line integral of the electric intensity F from 1 to 2 along any path in which there is no contact electromotive force or magnetically induced electromotive force.

As noted in Article 142, the electric intensity at any point P due to an unvarying equivalent charge q' at a distance x from P is

$$F = \frac{q'}{k_s x^2}$$

where k_s is dielectric permeability of free space. When there is no varying magnetic field in, or surrounding, the region in question this is also equal to the resultant electric intensity at P.

The drop of electric potential from any point 1 to any other point 2, due to a single unvarying equivalent charge q' is therefore equal to the line integral of $\frac{q'}{k_s x^2}$ from 1 to 2. By employing the same method of analysis as used in Article 107, it may readily be seen that this line integral has the value

$$v_{12} = \frac{q'}{k_s x_1} - \frac{q'}{k_s x_2} \tag{23}$$

where x_1 is the distance of the point 1 from the charge q' and x_2 is the distance of the point 2 from this charge.

When the point 2 is at an infinite distance from q' the drop of potential from 1 to 2 is simply

$$v_1 = \frac{q'}{k_s x_1} \tag{24}$$

The quantity $\frac{q'}{k_s x_1}$ is called the "absolute electric potential"

at the point 1 due to the unvarying equivalent charge q'; it is equal to the work done by a mechanical force, equal to the electric intensity F, in moving a particle from the point 1 to infinity.

Comparing (24) and (23) it is seen that the difference of electric potential between the two points 1 and 2 is equal to the difference of their absolute potentials. Since absolute electric potential is the ratio of work to charge, and since both work and charge are scalar quantities, absolute electric potential is also a scalar quantity. Hence the resultant absolute electric potential at any point due to any number of unvarying electric charges is the algebraic sum of the absolute potentials due to each charge separately. Therefore the absolute potential at any point P due to a charged surface of any kind is

$$v = \frac{1}{k_s}\int_S \frac{\sigma ds}{x} \tag{25}$$

where ds represents any elementary area of the surface, σ the surface density of the equivalent charge at ds, and x the distance of the point P from ds.

Since absolute potential is a scalar quantity, it is often more convenient, instead of calculating directly the resultant electric intensity at a given point due to a given distribution of charges, to find first the absolute potential v at this point, and then determine the electric intensity F from the value of v. The relation between F and v is exactly the same as that between the magnetizing force H, due to a given distribution of magnetic poles, and the absolute magnetic potential v due to these poles (see Article 107), viz.,

$$F \cos \theta = - \frac{dv}{dl} \tag{26}$$

where dl is an elementary length taken in any direction at the point under consideration, and $F \cos \theta$ is the component of the electric intensity in this direction.

XIII

ELECTROSTATIC CAPACITY

150. Electric Condensers.—Any two conductors so arranged that a difference of electric potential may be established between them are said to form an "electric condenser." The simplest form of condenser consists of two parallel plates separated by a dielectric (*e.g.*, air or other insulator). In Fig. 108 is shown such a condenser C connected in series with a switch S, battery B, and ballistic galvanometer G. The conductors which form a condenser may be of any shape, *e.g.*, cylinders, spheres, etc., but

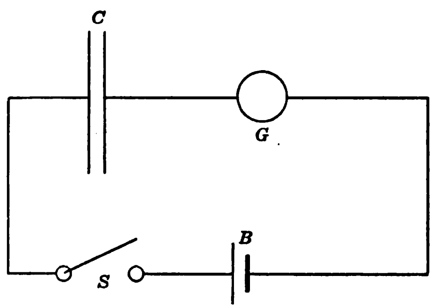


FIG. 108.

it is common practice to speak of the two conductors as the "plates" of the condenser.

Experiment shows that when the two plates of a condenser, originally insulated from all other conductors, are connected by wires to a source of electromotive force, these plates become equally and oppositely charged. This charge is due to the flow of electricity from one plate to the other, which flow ceases when the difference of potential V between the plates is equal to the electromotive force E of the battery.

When the two plates of a condenser have been charged in this manner, the plates will remain charged even when the wires connecting them to the battery are removed, provided they are not touched by any other conductor, and provided the dielectric of the condenser is a perfect insulator. Actually, however, since no dielectric is a perfect insulator, the charges on the plates will

gradually disappear, *i.e.*, there will be a flow of electricity through the dielectric from one plate to the other.

Similarly, when the two plates of a condenser which has been charged as above described are connected by a wire, a momentary current will flow through this wire, and the charges on the two plates will disappear (provided there is no electromotive force in the wire). Due to the relatively low resistance of a wire compared with the insulation resistance of the dielectric, the discharge of the condenser in this case usually takes place in a very small fraction of a second (see Article 157).

The quantity of electricity which flows from one plate to the other, through the connecting wires, when a condenser is charged or discharged, may be readily measured by means of a calibrated ballistic galvanometer (see Article 110). A ballistic galvanometer therefore serves as a means of measuring the *real* charge at the surface of separation between a conductor and a dielectric. The charge measured by a ballistic galvanometer does not include the apparent charge at such a surface, for this charge can never leave this surface, and therefore cannot flow through the galvanometer.

151. Electrostatic Capacity.—The numerical value of the charge acquired by either plate of a condenser when a difference of electric potential is established between these plates is found to depend (1) upon the value of this difference of potential, (2) upon the shape and extent of the surface of the plates, (3) upon the distance apart of the two plates, and (4) upon the nature of the dielectric between the plates. For a given condenser of fixed dimensions, the charge taken by either plate is found to be directly proportional to the difference of potential which is established between them, at least to a close approximation.

The ratio of the numerical value of the real charge taken by either plate of a condenser, when the two plates are equally and oppositely charged, to the numerical value of the potential difference between them, is called the electrostatic capacity of the condenser. The capacity of a condenser is usually represented by the symbol C. Hence, calling q the numerical value of the charge acquired by either plate when a difference of potential v is established between these plates, then

$$C = \frac{q}{v} \tag{1}$$

Strictly speaking, this definition applies only when there is no varying magnetic field between the two plates. A more general definition, which is always applicable, is that the capacity of a condenser is equal to the numerical value of the real charge q taken by either plate, when the two plates are equally and oppositely charged, divided by the line integral of the electric intensity F along any path from one plate to the other. When there is no appreciable varying magnetic field between the plates, which is practically always the case, this line integral is equal to the difference of electric potential between the plates (see Article 142).

The mathematical statement of this general definition is that the capacity of a condenser is

$$C = \frac{q}{\int F dl} \tag{1a}$$

where dl is any elementary length in a line of electric force connecting the two plates, F is the electric intensity at dl, and the integral is taken along this line of force from one plate to the other. As before, q is the numerical value of the charge on either plate.

Since a real charge q gives rise to 4π lines of electric force, the capacity of a condenser may also be defined as $\frac{1}{4\pi}$ times the quotient of the electric flux ψ from one plate to another divided by the line integral of the electric intensity from one plate to the other, viz.,

$$C = \frac{1}{4\pi} \frac{\psi}{\int F dl} \tag{1b}$$

The last expression for capacity may also be looked upon as defining the electrostatic capacity of *any portion of a dielectric* included between two equipotential surfaces and bounded laterally by lines of force. Comparing equation (1*b*) with equation (26*a*) of Article 92, it is evident that, with the exception of the factor 4π, the electrostatic capacity of a dielectric bears a relation to the electric flux and electric intensity similar to that between magnetic permeance (reciprocal of magnetic reluctance), magnetic flux and magnetizing force.

From the fact that the density of the conduction current at any point in a dielectric is γF, where γ is the conductivity of the dielectric, and that the density of the electric flux at this

point is kF, where k is the dielectric permeability of the dielectric, it may be readily shown that the ratio of the capacity of a condenser to the conductance of the dielectric between its plates is $\frac{k}{4\pi\gamma}$. Hence the formulas for the capacity and conductance of the dielectric between the plates of any shape or size of condenser differ only by a constant coefficient. That is, if g is the conductance of the dielectric between the plates of a condenser, no matter what the size, shape or arrangement of these plates, the capacity of this condenser is

$$C = \frac{kg}{4\pi\gamma} \tag{2}$$

The practical unit of electrostatic capacity is called the "farad." That is, when q is expressed in coulombs and v in volts, the value of C given by equation (1) is said to be so many farads. The c.g.s. electrostatic unit of capacity is called the "statfarad," and the c.g.s. electromagnetic unit the "abfarad." A capacity equal to one-millionth of a farad is called a "microfarad," and is the unit most commonly employed. These various units are related as follows:

$$1 \text{ farad} = 10^6 \text{ microfarads}$$
$$1 \text{ statfarad} = \tfrac{1}{9} \times 10^{-5} \text{ microfarads}$$
$$1 \text{ abfarad} = 10^{15} \text{ microfarads.}$$

Problem 1.—Two conductors of fixed size and shape are held in a fixed relative position by insulating supports. These two conductors are completely immersed in a salt solution whose specific resistance is 1.5 ohm-centimeters. It is found that when a current of 10 amperes is established through this solution, using the two conductors as electrodes, the total potential drop through the electrolyte is 0.2 volt. What is the electrostatic capacity of the condenser formed by these two conductors when separated by air?

Answer.—5.96 statfarads.

152. Calculation of the Capacity of Simple Condensers.—From the defining equations given in the preceding article, the capacity of certain simple forms of condensers may be readily calculated.

(*a*) *Parallel Plate Condenser.*—Let S be the area of each plate (one surface only) in square centimeters; let d be the distance between the two plates in centimeters, and let k be the dielectric permeability of the medium, assumed uniform, between the two plates. From the condition that there can be no electric field within the substance of either plate, it may readily be shown

that, when the plates are close together, the charge on each is uniformly distributed over that surface of the plate which is opposite the other plate (*i.e.*, over the "inside" surface of the plate), except for points near the edges of the plates (see Problem 2, Article 142).

Moreover, since the lines of force must be perpendicular to the surface of each plate, they will be practically straight and uniformly distributed, except near the edges of the plates. In short, the distribution of the electric lines of force between two parallel plates which are very close together is exactly the same as the distribution of the magnetic lines of force in a narrow air-gap between two iron cores, when the end faces of these cores are parallel to each other.

Let q be the charge on either plate. Then the total number of lines of electric force in the dielectric between the two plates is $4\pi q$. Assuming all these lines to be straight, parallel and uniformly distributed, the electric flux density in the dielectric between the two plates is

$$D = \frac{4\pi q}{S}$$

and the electric intensity at each point of the dielectric is

$$F = \frac{D}{k} = \frac{4\pi q}{kS}$$

The line integral of F from one plate to the other is simply Fd. Whence the difference of potential between the two plates is

$$v = \frac{4\pi qd}{kS} \tag{3}$$

The capacity of the condenser is therefore

$$C = \frac{kS}{4\pi d} \tag{4}$$

Note that the conductance of the dielectric between the two parallel plates is $g = \frac{\gamma S}{d}$, and compare with equation (2).

A parallel plate condenser consisting of but a single pair of plates has only a very small capacity, even when the plates are very close together. However, since the plates of a condenser may be made relatively thin, a multiple plate condenser, con-

structed as indicated in Fig. 109, may be made to occupy a relatively small volume and yet have a relatively high capacity. In such a multiple plate condenser *both* sides of each plate, except the two outside plates, become charged when a potential differ-

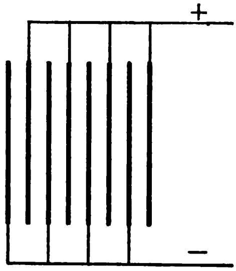


Fig. 109.—Parallel-plate condenser.

ence is established between the two sets of plates. Hence, calling N the total number of plates in the condenser, S the area of one surface of each plate in square centimeters, d the thickness of the dielectric between the plates in centimeters, and k the dielectric permeability of this dielectric, the capacity of this condenser is

$$C = (N - 1)\frac{kS}{4\pi d} \tag{4a}$$

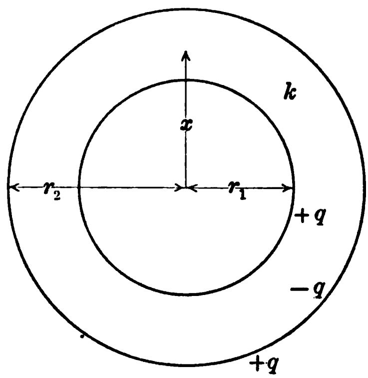


Fig. 110.—Spherical condenser.

(*b*) *Spherical Condenser.*—Another simple form of condenser consists of two concentric spherical shells, the space between which is filled with a uniform dielectric. Let the shells have radii of r_1 and r_2 centimeters and let the dielectric permeability of the dielectric between the two shells be k. A charge q units

given the inside sphere will induce a charge of $-q$ units on the inside surface of the outer shell, and a charge of $+q$ units on the outside surface of this shell and whatever other conductors may be connected to it. This outside charge $+q$, however, will have no effect on the electric intensity inside this shell (see Article 147).

From symmetry, the lines of electric force are radial lines normal to the surfaces of two spheres. The total number coming out from the charge q is $4\pi q$. Hence the electric flux density at any point P in the dielectric is

$$D = \frac{4\pi q}{4\pi x^2} = \frac{q}{x^2}$$

where x is the distance of P from the center of the spheres. Hence the field intensity at P is $F = \frac{D}{k} = \frac{q}{kx^2}$. Therefore the difference of potential between the two spheres is

$$v = \int_{r_1}^{r_2} \frac{q}{kx^2}\, dx = \frac{q}{k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Hence the capacity of this condenser is

$$C = \frac{kr_1r_2}{r_2 - r_1} \tag{5}$$

(c) *Coaxial Cylinders.*—Exactly similar reasoning applied to the case of two coaxial cylinders which are so long in comparison with their diameters that the lines of electric force going out the ends may be neglected, shows that the electric flux density at any point P in the dielectric between the two cylinders is $D = \frac{4\pi q}{2\pi x} = \frac{2q}{x}$, where q is the *charge per unit length* of the condenser, and x the distance of P from the center of the cylinders. Hence the electric intensity at P is $F = \frac{2q}{kx}$. Therefore the difference of potential between the two cylinders is

$$v = \int_{r_1}^{r_2} \frac{2q}{kx}\, dx = \frac{2q}{k} \log_e \frac{r_2}{r_1}$$

where r_1 is the outside radius of the inside cylinder, and r_2 the inside radius of the outside cylinder. Hence the capacity *per centimeter* of the condenser formed by two long coaxial cylinders is

$$C = \frac{k}{2 \log_e \frac{r^2}{r_1}} \tag{6}$$

Compare with the formula, equation (18), Article 65, for the insulation resistance of a single conductor cable.

This formula gives the capacity of a single conductor cable enclosed in a lead sheath, when r_1 is taken as the radius of the conductor and r_2 the inside radius of the sheath. For practical calculations it may be written

$$C = \frac{0.007354K}{\log_{10}\frac{r_2}{r_1}} \quad \text{microfarads per 1000 feet} \qquad (6a)$$

or

$$C = \frac{0.03883K}{\log_{10}\frac{r_2}{r_1}} \quad \text{microfarads per mile} \qquad (6b)$$

where K is the specific inductive capacity of the insulation.

(*d*) *Two Parallel Cylinders.*—Consider next the case of two parallel cylinders whose axes are D centimeters apart, *e.g.*, two

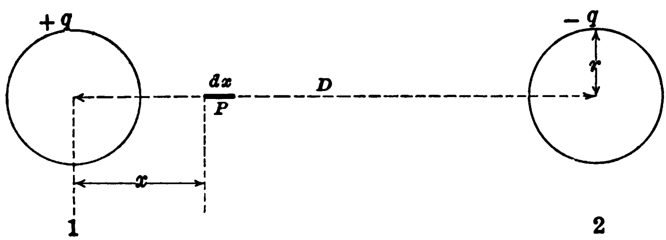


FIG. 111.—Two parallel wires.

parallel wires of a transmission line. Let r be radius of each cylinder (or wire) in centimeters. An approximate formula for the capacity per unit length of the condenser formed by two such cylinders may be readily derived, when they are long compared to their distance apart, and when their distance apart is large in comparison with their diameters. Under these conditions the charge on each cylinder may be assumed uniformly distributed over its surface.

On this assumption, the electric intensity at any point P on the line joining the centers of the two cylinders, due to the charge on No. 1, is $F_1 = \frac{2q}{kx}$, and the electric intensity at P

in the same direction, due to the charge $-q$ on No. 2 is $F_2 = \frac{2q}{k(D-x)}$. Therefore the total intensity at P is

$$F = \frac{2q}{k}\left[\frac{1}{x} + \frac{1}{D-x}\right]$$

Hence the difference of potential between the two cylinders is

$$v = \frac{2q}{k}\int_r^{D-r}\left[\frac{1}{x} + \frac{1}{D-x}\right]dx = \frac{4q}{k}\log_e \frac{D-r}{r}$$

Since by hypothesis D is large in comparison with r, the term $(D - r)$ may be taken equal to D. Hence the capacity per centimeter of the condenser formed by the two parallel cylinders is

$$C = \frac{k}{4 \log_e \frac{D}{r}} \tag{7}$$

For practical calculations this formula may be written

$$C = \frac{0.003677K}{\log_{10} \frac{D}{r}} \quad \text{microfarads per 1000 feet} \tag{7a}$$

or

$$C = \frac{0.01941K}{\log_{10} \frac{D}{r}} \quad \text{microfarads per mile} \tag{7b}$$

where K is the specific inductive capacity of the surrounding medium.

In the deduction of equation (7) it was assumed that the charges on the two cylinders, or wires, are uniformly distributed. This assumption, though permissible when the wires are far apart, gives entirely erroneous results when the wires are close together. When the actual distribution of the charges on the two wires is taken into account, the correct expression for the capacity of the condenser formed by them is

$$C = \frac{k}{4\cosh^{-1}\left(\frac{D}{2r}\right)} \quad \text{abfarads per centimeter} \tag{8}$$

$$= \frac{8.467 \times 10^{-3}K}{\cosh^{-1}\left(\frac{D}{2r}\right)} \quad \text{microfarads per 1000 feet} \tag{8a}$$

$$= \frac{0.0447K}{\cosh^{-1}\left(\frac{D}{2r}\right)} \quad \text{microfarads per mile} \tag{8b}$$

(See Alex Russell, *Alternating Currents*, Vol. 1, p. 99 and Pender and Osborne, *Elec. World*, Vol. 56, p. 667.) When the distance apart of the wires is 10 or more times their diameter, the approximate formulas (7) to (7*b*) are in error by less than 0.1 per cent.

In the deduction of all the formulas in this section it is assumed that there is in the vicinity of the two conductors which form the condenser under consideration no other conductors on which electric charges are induced. When the conductors in the vicinity of the conductors which form the two plates of a condenser are of the same order of magnitude as the latter, and their distance away is of the same order of magnitude as the distance apart of these two conductors, their presence may produce an appreciable effect on the capacity of the latter. For example, the capacity of the condenser formed by two parallel insulated wires enclosed in a lead sheath is appreciably affected by the presence of the sheath. For a discussion of the electrostatic capacity of multiple-conductor cables and various arrangements of overhead wires, see the article on *Capacity and Charging Currents* in Pender's *Handbook for Electrical Engineers*.

Problem 2.—What must be the area of each plate of a two-plate parallel plate condenser in order that this condenser have a capacity of 1 farad, if the distance between the plates is 0.01 inch and the dielectric has a specific inductive capacity of 2.5.

Answer.—4.42 square miles. (NOTE.—A parallel plate condenser of such dimensions is of course impracticable.)

Problem 3.—(*a*) Assuming the area of the conducting plates and dielectric sheets in a multiple-plate condenser to have the same area, what must be the volume of a condenser of 1 farad capacity if the conducting plates are 5 mils thick and the dielectric is paper 10 mils thick, having a specific inductive capacity of 2.5. (*b*) If this condenser is made in the shape of a cube what must be the length of each edge of this cube? (*c*) How many conducting sheets would there be in such a condenser? (*d*) What proportion of the volume of this condenser would have a capacity of 1 microfarad? (NOTE.—To avoid excessive leakage at the edge of the plates, the dielectric sheets must actually be made of greater area than the conducting plates, and consequently the volume of the condenser will be somewhat greater than that here calculated.)

Answer.—(*a*) 154,000 cubic feet. (*b*) 53.6 feet. (*c*) 43,000 plates. (*d*) $\frac{1}{1,000,000}$ of the volume of the 1-farad condenser, or 0.154 cubic feet, or 266 cubic inches. Such a condenser made in the form of a cube would be 6.43 inches on each edge. (NOTE.—A 2-microfarad condenser having a

volume of less than 10 cubic inches is actually used in telephone practice. The paper and foil used in this condenser are extremely thin (about 1 mil). In spite of the thinness of the insulation between plates, these condensers will withstand a potential difference of about 1000 volts.

Problem 4.—A certain overhead transmission line, consisting of two No. 0000 A. W. G. stranded copper wires, spaced 6 feet between their centers, is 100 miles long. The diameter of each wire, which may be considered a cylinder, is 0.528 inch.

(*a*) What is the capacity of the condenser formed by these two wires? (*b*) If a difference of potential of 100,000 volts is established between these wires, what will be the charge on each? (*c*) If this difference of potential is established in $\frac{1}{240}$ of a second, what will be the average rate of change of the charge on each wire during this interval? (*d*) What will be the average value of the electric current which flows from one wire to the other through the source of electromotive force which establishes this difference of potential? (*e*) Will there be any flow of electricity through the air which separates the wires? (*f*) Will there be a displacement current through the air while the potential difference is being established? (*g*) If so, what will be the average value of this displacement current?

Answer.—(*a*) 0.797 microfarads. (*b*) 0.0797 coulombs. (*c*) 19.1 coulombs per second. (*d*) 19.1 amperes. (*e*) No. (*f*) Yes, as long as the potential difference is varying. (*g*) 19.1 amperes, the same as the conduction current through the source of e.m.f. (NOTE.—Tables of capacities of parallel wires for various sizes of wires and various distances between their centers will be found in any electrical engineers handbook. Some of these tables give "capacity to neutral." The capacity of the condenser formed by two wires is one-half the capacity to neutral. See the article on *Capacity and Charging Current* in Pender's *Handbook for Electrical Engineers.*

Problem 5.—A lead-covered cable is made of a No. 00 B. & S. wire surrounded by a layer of rubber 0.25 inch thick, which is in turn surrounded by a layer of gutta-percha 0.25 inch thick, the whole being encased in the lead sheath. The specific inductive capacity of the rubber is 2.2 and specific inductive capacity of the gutta-percha is 4.5. What is the ratio of the potential drop through the rubber to that through the gutta-percha when a given difference of potential is established between the wire and the sheath? The resistance of each dielectric is to be considered infinite. The diameter of a No. 00 wire is 0.365 inch.

Answer.—3.87.

153. Condensers in Series and in Parallel.—When two or more condensers are connected in series (Fig. 112), and equal and opposite charges q and q' are given the two outside plates of the series, charges equal in numerical value will be induced on each of the other plates, as indicated in the figure. Hence if the condensers have capacities C_1, C_2, C_3, etc., the total potential drop through all the condensers is

$$v = \frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3} + - - - - = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + - - -\right)$$

Hence the equivalent capacity of any number of condensers in series is C_s where

$$\frac{1}{C_s} = \frac{v}{q} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + - - - \tag{9}$$

When the condensers are connected in parallel (Fig. 113), the drop of potential across each condenser is the same, and there-

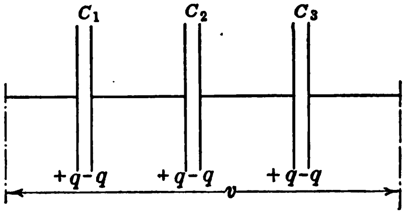


FIG. 112.—Condensers in series.

fore the total positive charge given all the condensers (equal to the total negative charge) is

$$q = (C_1 + C_2 + C_3 + - - -)\, v$$

Hence the equivalent capacity of any number of condensers in parallel is

$$C_p = \frac{q}{v} = C_1 + C_2 + C_3 + - - - \tag{10}$$

Compare with resistances in series and parallel, Article 98.

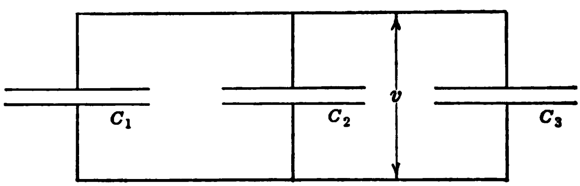


FIG. 113.—Condensers in parallel.

Problem 6.—If there are available for a certain test three condensers whose capacities are 1, 2 and 3 microfarads respectively, what capacities can be obtained by suitably connecting one or more of these condensers? Indicate by a sketch the connections for each of these capacities.

Answer.—$\frac{6}{11}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, 1, $1\frac{1}{5}$, $1\frac{1}{3}$, $1\frac{1}{2}$, 2, $2\frac{1}{5}$, $2\frac{3}{4}$, 3, $3\frac{2}{3}$, 4, 5, 6 microfarads.

Problem 7.—Three condensers, *A*, *B*, and *C*, the capacities of which are 25, 20, and 15 microfarads respectively, are connected in series. The potential drop across *B* is 100 volts.

(*a*) What is the potential drop across *A*? (*b*) Across *C*? (*c*) What is the total potential drop across the three in series? (*d*) What is the charge on each condenser? (*e*) What is the capacity of the three condensers in series?

Answer.—(*a*) 80 volts. (*b*) 133.3 volts. (*c*) 313.3 volts. (*d*) 0.002 coulomb. (*e*) 6.38 microfarads.

154. Measurement of Electrostatic Capacity and Dielectric Constant.—By definition (Article 151) the capacity of a condenser is the ratio of the charge on either plate to the potential difference between the two plates. Hence the capacity of a condenser may be determined by connecting it in series with a source of electromotive force, ballistic galvanometer and switch, as indicated in Fig. 108.

If the condenser is initially uncharged (as can be assured by connecting its two plates momentarily with a wire), the quantity of electricity which flows through the galvanometer G when the switch S is closed is numerically equal to the charge q given either plate. This quantity of electricity is approximately proportional to the first swing of the galvanometer (see Article 86).

In order to determine the exact relation between the first swing of the galvanometer and the quantity of electricity discharged through it, it is necessary first to calibrate the galvanometer, either by means of a standard solenoid (see Article 110) or by using in place of the condenser C a standard condenser of known capacity. Any form of condenser whose capacity can be calculated (see Article 152), to sufficient degree of accuracy for the purpose in hand, may be used as such a primary standard of capacity.

The potential difference between the plates of the condenser when charged may be measured by means of a voltmeter or potentiometer. The capacity is then found by taking the ratio of the quantity of electricity indicated by the galvanometer to the potential difference between the plates.

There are other and more convenient methods of measuring the capacity of a condenser, in which an alternating potential difference is employed. See Article 176, and for further details any text-book on electric measurements.

Since the capacity of a given condenser is directly proportional to the dielectric permeability of the dielectric between its plates,

the ratio of the capacity of a given condenser with a given dielectric between its plates, to its capacity when there is vacuum between its plates, is equal to the specific inductive capacity of the given dielectric (see Article 139).

Hence the name "specific inductive capacity" for the ratio of the dielectric permeability of a dielectric to the dielectric permeability of free space.

Experiment shows that a given condenser with air between its plates has practically the same capacity when the air is exhausted, leaving a vacuum between the plates. Hence the dielectric constant of air is practically unity, and therefore air instead of a vacuum may be used as the dielectric of reference in determining the specific inductive capacity of any other substance.

155. Electric Absorption and Dielectric Hysteresis.—Experiment shows that when a given difference of potential is established between the plates of a condenser which has a solid dielectric, the charge taken by the condenser depends upon the length of time this potential difference is maintained Again, when such a charged condenser is discharged, by connecting its plates momentarily with a conductor, and this connection is then broken, the plates at first appear to be entirely discharged, but after a few seconds a charge again appears on them, resulting in the reëstablishment of a difference of potential (less than in the first case) between the plates.

In short, a solid dielectric apparently absorbs a certain amount of charge which it gives up only after a considerable lapse of time. Hence the name "electric absorption" for this phenomenon. The charge which appears on the plates of the condenser after the first discharge is called the "residual" charge. The absorption of a dielectric is apparently due to impurities in it, or to lack of homogeneity. It is greatest in substance like mica and glass; it is doubtful if absorption occurs at all in absolutely homogeneous substances.

A phenomenon closely associated with electric absorption is the fact that when the electric field in a heterogeneous dielectric is caused to vary rapidly, an amount of heat is dissipated in the dielectric greatly in excess of that which can be accounted for in terms of its leakage resistance as determined by direct-current measurements. For example, when the resistance of the dielectric of a condenser to a *direct current* is r ohms, a constant poten-

tial difference of V volts impressed across its terminals will cause a current of $I = \frac{V}{r}$ amperes to flow through the dielectric, and heat will be developed in it at a rate of $rI^2 = \frac{V^2}{r}$ joules per second (or watts). This heat power may also be written gV^2, where $g = \frac{1}{r}$ is the conductance of the dielectric to a direct current. This conductance is usually called the "leakage" conductance of the condenser to a direct current, or its direct-current "leakance," and the current $I = gV$ is called the "leakage current" corresponding to the constant potential difference V.

When an alternating potential difference of the same r.m.s. value (see Article 165) is established across the condenser, it is found that, when the dielectric of the condenser possesses the property of electric absorption, the heat energy dissipated in the condenser is greater than gV^2, where g is its direct-current conductance. This increase in the heat energy may be due to an actual increase in the conductance of the dielectric with the speed of variation of the field, or may be due to a phenomenon analogous to magnetic hysteresis, *i.e.*, to a lag of the electric flux density behind the electric intensity. Whatever may be the cause of this extra loss of power for rapidly varying fields, it is generally described as the loss due to "dielectric hysteresis." The heat developed is in many cases quite appreciable. See the article on *Condensers, Electric,* in Pender's *Handbook for Electrical Engineers.*

156. Charging Current and Leakage Current of a Condenser. —When a condenser is charged by connecting it to a source of electromotive force, the current i which flows up to the positive plate of the condenser, through the connecting wire, is equal to the rate i_g at which electricity flows away from this plate through the dielectric, plus the rate $\frac{dq}{dt}$ at which electricity accumulates on this plate, viz.,

$$i = i_g + \frac{dq}{dt} \tag{11}$$

The rate i_g at which electricity flows away from the plate through the dielectric is called the "leakage" current through the dielectric, which in turn is equal to the product of the leakage conductance g by the drop of electric potential v through the

dielectric from the positive to the negative plate of the condenser, viz.,

$$i_g = gv \tag{12}$$

The value of the leakage conductance g, in the case of a rapidly varying potential difference v, usually depends upon the *rate* of variation of this potential difference (see Article 155).

That portion of the current in the connecting wires which is equal to rate at which electricity accumulates on the positive plate of the condenser, which in turn is equal to the rate at which positive electricity leaves (or negative electricity accumulates at) the negative plate, is called the "charging current" of the condenser. That is, calling dq the increase in the value of the charge on the positive plate in time dt, the charging current of the condenser during this interval is

$$i_c = \frac{dq}{dt} \tag{13}$$

Since the charge on either plate of a condenser at any instant is numerically equal to the product of the capacity C of the con-

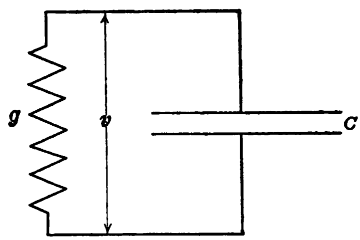


FIG. 114.—Diagram of leaky condenser.

denser by the potential difference v between its plates, the charging current may also be written

$$i_c = \frac{d}{dt}(Cv) \tag{13a}$$

When the capacity of the condenser is a constant, as is usually the case in practice, unless the phenomenon of dielectric hysteresis is pronounced, this relation may be written

$$i_c = C\frac{dv}{dt} \tag{13b}$$

The total current in the connecting wires is then, from equation (11),

$$i = gv + C\frac{dv}{dt} \tag{14}$$

From this relation it follows, as may be readily seen by applying Kirchhoff's first law (Article 29) to the circuit represented in Fig. 114, that a leaky condenser of capacity C and leakage conductance g is equivalent to a condenser of capacity C, without leakage, in parallel with a non-inductive resistance whose conductance is g.

Equation (14) should also be compared with equation (28*a*), Article 126, viz.,

$$v = ri + L\frac{di}{dt} \tag{15}$$

for the relation between the potential drop v through a coil whose resistance is r and whose self-inductance is L, when a varying current i is flowing through the coil.

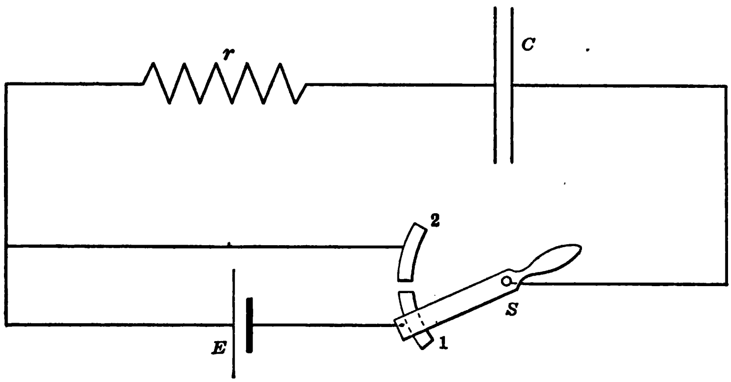


FIG. 115.

Note particularly that when the voltage across a condenser is *constant*, there is no charging current, just as when the current in a coil is not varying there is no back electromotive force of self-induction. When the voltage across a condenser is constant, the only flow of electricity to the plates is the leakage current which flows up to the positive plate, through the dielectric, and away from the negative plate. Similarly, when the current in a coil is constant, the only drop of potential through it is the drop due to its resistance.

157. Charge and Discharge of a Condenser Through a Resistance.—Fig. 115 represents diagrammatically a condenser of capacity C and zero leakage conductance in series with a non-inductive resistance r, a battery whose electromotive force is E, and a switch S. The resistance r designates the *total* resist-

ance of the circuit external to the condenser. (The resistance of the dielectric of the condenser, on the assumption of no leakage, is infinite.) The total self-inductance of the circuit is assumed to be negligible.

Let the blade of the switch S be initially on the contact 2, thus short-circuiting the resistance and condenser, and insuring that the condenser is discharged. Now let the blade of switch be moved to contact 1, connecting the battery in circuit. A positive charge will immediately begin to flow from the negative to the positive plate of the condenser through the battery and resistance r (or a negative charge will flow in the opposite direction), the battery acting, so to speak, as a "pump" which forces the electricity from one plate to the other.

As the charge on the condenser increases, the potential difference between its plates increases, until this difference of potential is equal to the electromotive force of the battery, corresponding to a final value of the charge equal to the capacity of the condenser multiplied by the battery electromotive force E, viz., to CE. At any instant t at which the charge on the condenser is less than this value, the potential drop through the condenser (from its positive to its negative plate) is

$$v = \frac{q}{C}$$

where q is the charge on it at time t.

This drop of potential through the condenser, plus the resistance drop in the external circuit, must at each instant be equal to the electromotive force of the battery. The current in the external circuit is equal to the charging current of the condenser, and is, therefore, equal to the rate of increase of the charge on the positive plate, viz., to $\frac{dq}{dt}$. Hence at any instant t the relation

$$E = \frac{q}{C} + r\frac{dq}{dt} \tag{16}$$

must hold. Compare with equation (28a), Article 126.

This equation is solved in identically the same manner as equation (28a), Article 126, giving for q at any instant t after the condenser is connected to the battery the value

$$q = CE(1 - \epsilon^{-\frac{t}{rC}}) \tag{17}$$

The physical interpretation of this equation is that the charge on the condenser reaches its steady value $Q = CE$ only after the time t measured from the instant that the electromotive force is impressed on the circuit has become sufficiently great to make the term $\epsilon^{-\frac{t}{rC}}$ sensibly equal to zero. When the resistance of the circuit through which the condenser is charged is only a few ohms, and C is of the order of a *microfarad*, the term $\epsilon^{-\frac{t}{rC}}$ becomes

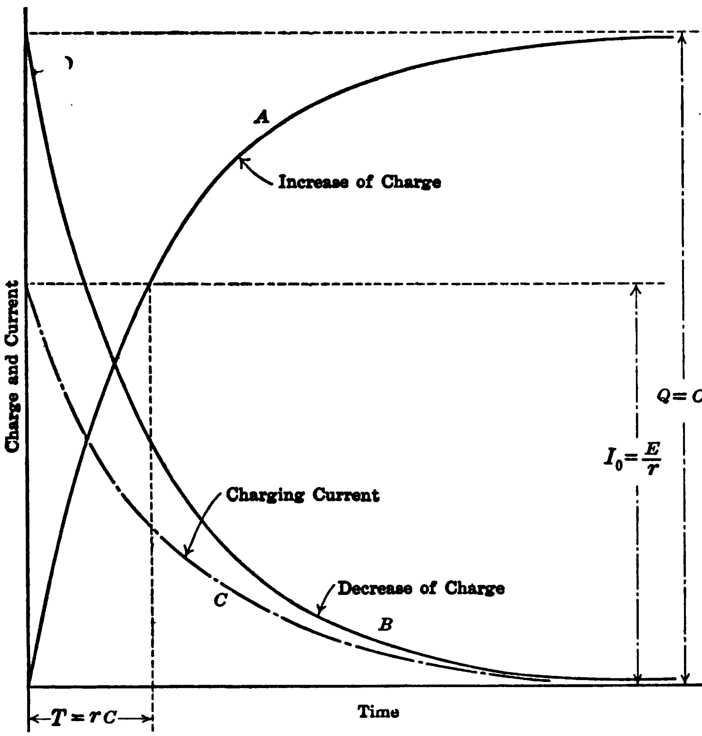


FIG. 116.—Charge and discharge of a condenser through a resistance.

practically zero for t equal to a small fraction of a second, and the condenser therefore becomes fully charged almost immediately after the circuit is closed. This is true only when the capacity is a constant, *i.e.*, when there is no electric absorption (see Article 155). When the phenomenon of electric absorption is pronounced, the charge may continue to increase for several seconds or even minutes after the electromotive force is impressed.

The time $T = rC$ required for the charge on the condenser

to reach $(1 - \epsilon^{-1})$, or 63.2 per cent., of its final value is called the time constant of the condenser and resistance. Compare with the time constant $\frac{L}{r}$ of a coil whose self-inductance is L and whose resistance is r.

The relation between the charge on the condenser and time given by equation (17) may be represented graphically by plotting the charge q as ordinates against time t as abscissas, as shown by the curve A in Fig. 116.

The current i through the battery and resistance r is the charging current, and its value is therefore, from equation (17),

$$i = \frac{dq}{dt} = \frac{E}{r}\epsilon^{-\frac{t}{rC}} \tag{18}$$

The physical interpretation of this equation is that the value of the current in the circuit at the instant after the switch S is closed, in order to charge the condenser, is equal to the *steady* current $\frac{E}{r}$ which would flow in the circuit were the condenser not present, and that as time goes on this current falls off, at first rapidly, and then more slowly, ultimately becoming zero when the condenser is fully charged. This is shown by curve C in Fig. 116.

From equation (18) it would appear that, upon closing the switch S, the current jumps instantaneously from zero (no current before switch is closed) to a finite value $\frac{E}{r}$. Actually, however, due to the inductance of the circuit, which can never be completely eliminated, the current rises continuously, although very rapidly, from zero to a maximum value (substantially equal to $\frac{E}{r}$ when the inductance is small), and then falls off as indicated by equation (18) and curve C. See the latter part of Article 158.

When the charge on the condenser at time $t = 0$ has a value Q_o, say, the solution of (16) is

$$q = CE(1 - \epsilon^{-\frac{t}{rC}}) + Q_o\epsilon^{-\frac{t}{rC}} \tag{19}$$

Compare with equation (29*a*), Article 126. In particular, when a condenser on which the charge is Q_o is short-circuited at a given instant (*e.g.*, by shifting the blade of the switch S in Fig.

115 from the contact 1 to the contact 2), the charge on the condenser t seconds after this change is made will be

$$q = Q_o \epsilon^{-\frac{t}{rC}} \tag{19a}$$

and the current at this instant will be

$$i = -\frac{Q_o}{rC} \epsilon^{-\frac{t}{rC}} \tag{20}$$

The minus sign in the last equation signifies that the direction of the current i is *away from* the positive plate.

The variation of charge during the discharge of the condenser, as given by equation (19*a*), is shown graphically by curve B in Fig. 116. The current during discharge varies in exactly the same manner as during charge (curve C), except that it is in the opposite direction, *i.e.*, through the resistance in the direction from the positive to the negative plate of the condenser.

Problem 8.—An electromotive force of 250 volts is impressed upon a circuit formed by a non-inductive resistance of 1000 ohms in series with a condenser of 50 microfarads. Plot to scale the charge on the condenser and the current in the resistance during the first fifth of a second after the electromotive force is impressed. Make calculations for t = 0.01, 0.03, 0.05, 0.1, 0.15 and 0.2.

Answer.—(*a*) One point on curve: $t = 0.1$, $q = 0.0108$ coulomb, $i = 0.0338$ ampere.

Problem 9.—A condenser whose capacity is 50 microfarads is charged to a certain voltage and the source of electromotive force is removed. The insulation resistance of the condenser is 1 megohm.

(*a*) How long will it take for the voltage across the condenser to fall to one-half its initial value? (Take note of Fig. 114.) (*b*) If the condenser has a capacity of only 1 microfarad and the same insulation resistance (1 megohm), how long will this time be? (*c*) What insulation resistance must a 1-microfarad condenser have in order that the charge on it will not decrease by more than 5 per cent. during the first minute after being disconnected from the source which charges it.

Answer.—(*a*) 34.7 seconds. (*b*) 0.69 second. (*c*) 1180 megohms.

158. Discharge of a Condenser Through an Inductance.—When a condenser is allowed to discharge through a coil which has an appreciable inductance L and a low resistance, the variation of the charge on the condenser and the current in the coil is entirely different from that which takes place when the condenser discharges through a non-inductive resistance. Both the charges on the plates and the current in the coil *oscillate*,

passing repeatedly through zero values and reversing in sign or direction, as may be readily seen by considering the conditions which must be satisfied in such a circuit at each instant.

In any practical case the oscillations are gradually damped out, due to the dissipation of energy as heat in the conductor which forms the coil and in the dielectric of the condenser, in exactly the same manner as the oscillations of a pendulum are damped out by friction. The analysis of the phenomena which take place is relatively simple when the resistance and leakance are assumed to be zero, and this ideal case will therefore be considered first.

Let v be the potential drop through the condenser at any instant, from the plate a to the plate b, as indicated in Fig. 117,

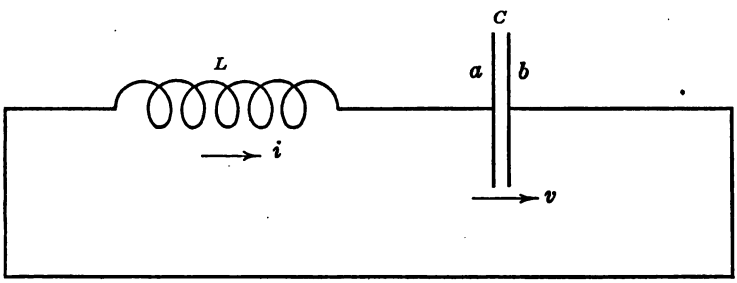


FIG. 117.—Coil and condenser in series.

and let i be the current in the coil at this instant, in the direction indicated in the figure. Let t designate time in seconds. The back electromotive force of self-induction in the coil at this instant is then $L\frac{di}{dt}$. Assuming the resistance of the coil to be zero, there will be no resistance drop through the coil, and therefore the resultant drop of potential through the coil in the direction of the current i, namely, around the circuit from the plate b to the plate a, is $L\frac{di}{dt}$. The drop of potential through the coil in the opposite direction, namely, from the plate a to the plate b is therefore $-L\frac{di}{dt}$. Since the drop of potential from one point to another can have at a given instant but a single value, it follows that at every instant $-L\frac{di}{dt}$ must be equal to the drop of potential v through the condenser, from the a plate to the b plate, viz.,

$$v = -L\frac{di}{dt} \tag{21}$$

Assuming the dielectric of the condenser to be a perfect insulator, and therefore no leakage current, the current i in the coil is equal to the charging current $\frac{dq}{dt}$ of the condenser, and the charge q on the a plate of the condenser at each instant is $q = Cv$. Whence

$$i = C\frac{dv}{dt} \tag{21a}$$

This substituted in equation (21) gives

$$\frac{d^2v}{dt^2} = -\frac{1}{LC}v \tag{21b}$$

as a relation which must be satisfied at each instant by the potential difference between the plates of the condenser.

The only relation between voltage and time which at every instant will satisfy this condition is

$$v = A\sin\left(\frac{t}{\sqrt{LC}} + \theta\right) \tag{22}$$

where A and θ are both constants. That equation (22) does satisfy the condition expressed by equation (21b) may be readily seen by differentiating (22) twice and substituting in (21b).

The value of the current i corresponding to the value of v given by equation (22) is found by substituting this value of v in (21a), which gives

$$i = \sqrt{\frac{C}{L}}A\cos\left(\frac{t}{\sqrt{LC}} + \theta\right) \tag{22a}$$

The values of the constants A and θ depend upon the values of the voltage v and current i at the instant of time taken as zero time. Consider first the case when at time $t = 0$ the voltage across the condenser has the value V, and the current in the coil is zero. These values substituted in equations (22) and (22a) give for A and θ the values $A = V$, and $\theta = \frac{\pi}{2}$. At any instant t seconds after the instant at which $v = V$ and $i = 0$, the voltage v and current i then have the values

$$v = V\cos\left(\frac{t}{\sqrt{LC}}\right) \tag{23}$$

$$i = -\sqrt{\frac{C}{L}}V\sin\left(\frac{t}{\sqrt{LC}}\right) \tag{23a}$$

In these expressions the quantity $\frac{t}{\sqrt{LC}}$ is in *radians*, not degrees. The corresponding angle in degrees is

$$\frac{57.3t}{\sqrt{LC}}$$

The significance of equations (23) and (23*a*) may best be seen by plotting these relations against time as abscissas, as shown in Fig. 118. The resulting curves are both "sine waves," reaching their maximum values and passing through their zero values at different times. The interval between the maximum value of v and the nearest maximum value of i is $\frac{\pi}{2}$ radians (or 90 degrees),

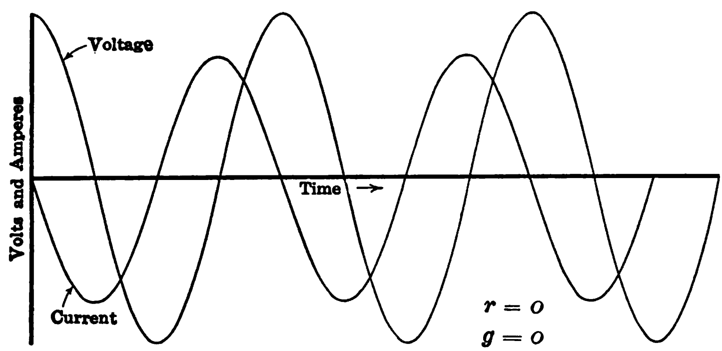


FIG. 118.—Discharge of a capacity through an inductance.

corresponding to a time interval of $\frac{\pi}{2}\sqrt{LC}$. This relationship is conveniently described by saying that the voltage and current differ in "phase" by $\frac{\pi}{2}$ radians, or 90 degrees (see Article 166).

As shown by the curves, both the current and voltage pass through a complete cycle of values (positive and negative) in an interval of time T determined by the relation $\frac{T}{\sqrt{LC}} = 2\pi$. This time interval T, namely, the time required for either the current or the voltage to pass through a complete cycle of values, is called the "natural period" of the circuit formed by the coil and condenser. This period is called the *natural* period of the circuit, since it is the period of the electric oscillations in it when the source which starts the oscillations (*e.g.*, the battery which

charges the condenser) is removed from the circuit. The natural period of a circuit formed by a condenser having zero leakage conductance and a coil of zero resistance is therefore

$$T = 2\pi\sqrt{LC} \qquad \text{seconds} \tag{24}$$

where L is the self-inductance of the coil and C is the capacity of the condenser. In this formula both L and C must be expressed in the *same* system of units.

The number of times per second that an oscillation passes through a complete cycle of values is called the "frequency" of the oscillation. The natural frequency of a circuit formed by a coil and a condenser whose resistance and leakance are respectively zero is therefore

$$f = \frac{1}{2\pi\sqrt{LC}} \qquad \text{cycles per second} \tag{24a}$$

Equations (23) and (23a) were deduced from the general relations expressed by (22) and (22a) on the assumption that at

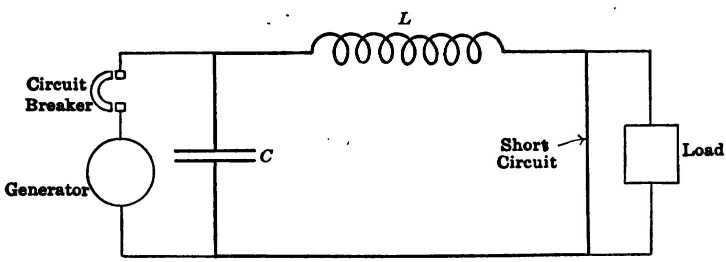


Fig. 119.

time $t = 0$ the voltage across the condenser is V and the current in the coil zero. Another case which is frequently realized in practice, at least to a first approximation, is when the current in the coil initially has a value I, say, and the voltage across the condenser is zero. This condition is approximately realized in the case of a transmission line which is suddenly short-circuited at the load end, and open-circuited (by the opening of a circuit breaker) at the generator end (see Fig. 119). As a first approximation the condenser formed by the line wires may be considered equivalent to a concentrated, or "lumped," capacity C at the sending end, as shown in Fig. 119.

Let I be the current in the line at the instant the circuit breaker opens. As a first approximation, the voltage across

the condenser may be assumed zero at this instant. Substituting $v = 0$ and $i = I$ corresponding to $t = 0$ in equations (22) and (22a), there results $A = I\sqrt{\frac{L}{C}}$ and $\theta = 0$. Whence, for these conditions (22) and (22a) become

$$v = \sqrt{\frac{L}{C}}\, I \sin\left(\frac{t}{\sqrt{LC}}\right) \tag{25}$$

$$i = I \cos\left(\frac{t}{\sqrt{LC}}\right) \tag{25a}$$

For example, consider the case of a line formed by two No. 0000 A. W. G. stranded copper wires, each 50 miles long, and spaced 6 feet between centers. The self-inductance of the loop formed by the two wires is then 0.193 henry, and the capacity of the condenser formed by the two wires is 0.40×10^{-6} farads. Let the current in the line, at the instant the circuit breaker opens, be 500 amperes. The natural frequency of the short-circuited line (under the assumptions above stated) is then

$$f = \frac{1}{2\pi\sqrt{0.193 \times 0.4 \times 10^{-6}}} = 573 \text{ cycles per second}$$

The maximum value of the current in the line will be 500 amperes, the initial value. The voltage between wires, however, will rise far above the normal voltage of the system, reaching a maximum value

$$\sqrt{\frac{L}{C}}\, I = \sqrt{\frac{0.193}{0.4 \times 10^{-6}}} \times 500 = 348{,}000 \text{ volts}$$

Hence, under the assumptions above made, the sudden opening of a current of 500 amperes in the given transmission line will set up an oscillation of current and voltage, each having a natural frequency of 573 cycles per second. The current will never be greater than the initial current, but the voltage between wires will oscillate between positive and negative maximum values equal to the relatively enormous value of 348,000 volts.

Actually, before the voltage reaches its first maximum value, one or more of the line insulators would be punctured, unless the line is equipped with "lightning arresters," in which case the discharge will be shunted through the arrester. The simplest form of such a device consists of a series of spark gaps shunted across the line. These gaps are adjusted so that they

will break down when the voltage between wires reaches a predetermined value, thus relieving the strain on the insulators. It is of interest to note that the protection of a line against lightning discharges is only one of the functions of a lightning arrester; such a device is likewise necessary to protect the line (and the apparatus connected therewith) against the abnormal voltage which may be produced when heavy loads are switched on or off the line.

As noted above, the relations deduced in this article are only approximately true for a transmission line. This is due to the fact that the capacity and inductance of the line are not "lumped," but distributed throughout its length. In particular, a transmission line has not only a fundamental natural frequency, but may also oscillate at higher frequencies which are multiples of the fundamental frequency. The exact formula for the natural fundamental frequency of a line short circuited at one end and open at the other is

$$f = \frac{1}{4\sqrt{LC}} \tag{26}$$

where L and C are the total inductance and capacity of the line. The fundamental natural frequency of the transmission line considered above is therefore 899 instead of 573 cycles per second. See the article on *Transmission Lines* in Pender's *Handbook for Electrical Engineers.*

A consideration of the energy relations involved in the electric oscillations here considered is extremely helpful in getting a physical conception of what takes place. Neglecting the dissipation of heat in the conductors and dielectric, the total energy of the system formed by the coil and condenser must remain *constant*, since no energy enters or leaves the system. When there is no voltage between the plates of the condenser, all the energy of the system is in the magnetic field produced by the current, and is equal to $\frac{1}{2} LI^2$, where I is the value of the current when the voltage across the condenser is zero. As shown by the equations and curves, this is also maximum value of the current.

When the current becomes zero, the voltage across the condenser reaches its maximum value V, and the energy of the magnetic field disappears as magnetic energy. This energy, however, is not "lost," but, from the principle of the conservation of energy, must be transferred to the condenser. In fact, as will be shown

in Article 159, an electric field in a dielectric represents a definite amount of stored energy, just as a magnetic field represents a definite amount of stored energy. In the particular case of a condenser of capacity C charged to a voltage V, this stored energy is $\frac{1}{2} CV^2$. Hence the relation between the maximum value of the current I in the coil and maximum value of the voltage $\overline{V}$ across the condenser must be such that

$$\frac{1}{2}LI^2 = \frac{1}{2}C\overline{V}^2$$

or

$$V = I\sqrt{\frac{L}{C}}$$

This relation is identical with that given by equation (25) for the maximum value of V $\left(\text{corresponding to } \frac{t}{\sqrt{LC}} = \frac{\pi}{2}\right)$.

The surging back and forth of the energy of the system formed by a coil and condenser is exactly analogous to the transformation of kinetic into potential energy, and *vice versa,* in a mechanical oscillating system formed by a weight hung from a spiral spring. See Problem 14, Article 18.

The relations thus far developed in this article are all deduced on the assumption that the resistance of the conductor which forms the coil and the leakage conductance of the dielectric are both zero. Actually, however, the resistance of the coil is always appreciable, and the leakance of the condenser, though usually negligible, is also appreciable in some instances. For the complete solution of the circuit consisting of a coil of resistance r and inductance L in series with a condenser of capacity C and leakance g see the article on *Transient Electric Phenomena* in Pender's *Handbook for Electrical Engineers.*

As shown in the article just referred to, when the resistance r and leakance g are appreciable, the frequency of the natural oscillation of the circuit formed by a coil and condenser is

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \left[\frac{1}{2}\left(\frac{r}{L} - \frac{g}{C}\right)\right]^2} \qquad (27)$$

This reduces to equation (24*a*) when r and g are both zero. When the term in the square brackets is greater than $\frac{1}{LC}$, the

natural frequency of the circuit becomes imaginary, *i.e.*, there is no oscillation of current and voltage. Neglecting the leakance g, this relation is equivalent to the condition that

$$r^2 > \frac{4L}{C} \tag{28}$$

In other words, when the resistance of the coil is greater than $2\sqrt{\frac{L}{C}}$, and the leakance of the condenser is negligible, there is no oscillation.

For the special case of a condenser of capacity C and leakance g, charged to a potential V_o, and then allowed to discharge through a coil of inductance L and resistance r, the value of v and i at any time t seconds after the oscillation begins are

$$v = V_o\epsilon^{-ut}\left(\cos \omega_o t + \frac{q}{\omega_o} \sin \omega_o t\right) \tag{29}$$

$$i = -\frac{V_o}{\omega_o L}\epsilon^{-ut} \sin \omega_o t \tag{29a}$$

(See equations (12) in the article in the Handbook just referred to.) The constant u in these equations, which constant may be called the "damping constant," or "attenuation constant," has the value

$$u = \frac{1}{2}\left(\frac{r}{L} + \frac{g}{C}\right) \tag{30}$$

The constant q, which may be called the "distortion constant," has the value

$$q = \frac{1}{2}\left(\frac{r}{L} - \frac{g}{C}\right) \tag{30a}$$

The constant ω_o, which may be called the "natural frequency constant," has the value

$$\omega_o = \sqrt{\frac{1}{LC} - q^2} \tag{30b}$$

The natural frequency is then

$$f_o = \frac{\omega_o}{2\pi} \tag{30c}$$

Equations (29) and (29a) are applicable only when the distortion constant q is less than $\frac{1}{\sqrt{LC}}$, for when $q > \frac{1}{\sqrt{LC}}$ the fre-

quency constant ω_o becomes imaginary. Under these conditions, namely; when

$$q > \frac{1}{\sqrt{LC}}$$

the values of v and i are given by the following relations, viz.,

$$v = V_o\epsilon^{-ut}\left(\cosh \omega' t + \frac{q}{\omega'} \sinh \omega' t\right) \tag{31}$$

$$i = -\frac{V_o}{\omega' L}\epsilon^{-ut} \sinh \omega' t \tag{31a}$$

where

$$\omega' = \sqrt{q^2 - \frac{1}{LC}} \tag{32}$$

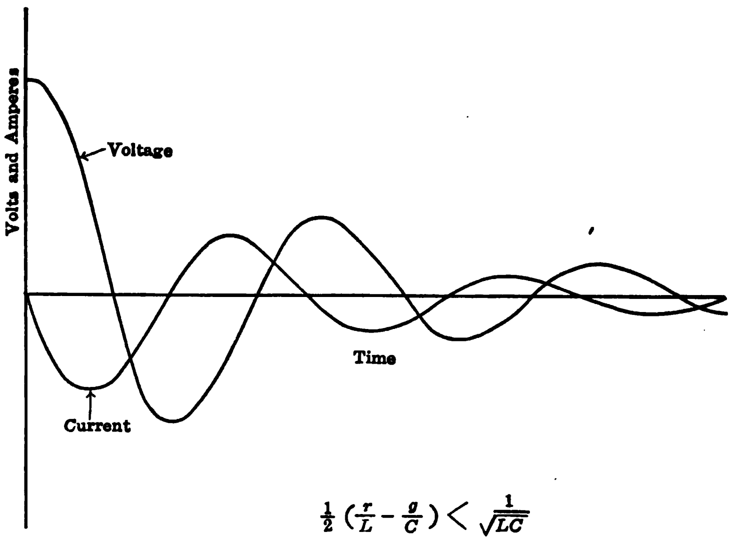


FIG. 120.—Discharge of a condenser through a coil.

and "cosh" and "sinh" stand respectively for the hyperbolic cosine and hyperbolic sine. The values of u and q are given by equations (30) and (30a).

The oscillation of the voltage and current represented by equations (29) and (29a) is shown graphically in Fig. 120. The variation with time in the voltage and current represented by equations (31) and (31a) is shown graphically in Fig. 121. These curves show clearly the damping effect of the dissipation of energy in the conductors in the circuit and in the dielectric

of the condenser. Note that when the distortion constant $q = \frac{1}{2}\left(\frac{r}{L} - \frac{g}{C}\right)$ is greater than $\frac{1}{\sqrt{LC}}$, there is no oscillation either of the voltage or current. The voltage falls to zero without reversing in sign, and the current rises to a maximum value and then falls off to zero also without reversing in direction.

Problem 10.—What must be the self-inductance of a coil which, when connected in series with a condenser of 50 microfarads, will have a natural frequency of 60 cycles per second? Neglect the resistance of the coil and the conductance of the condenser.

Answer.—0.141 henry.

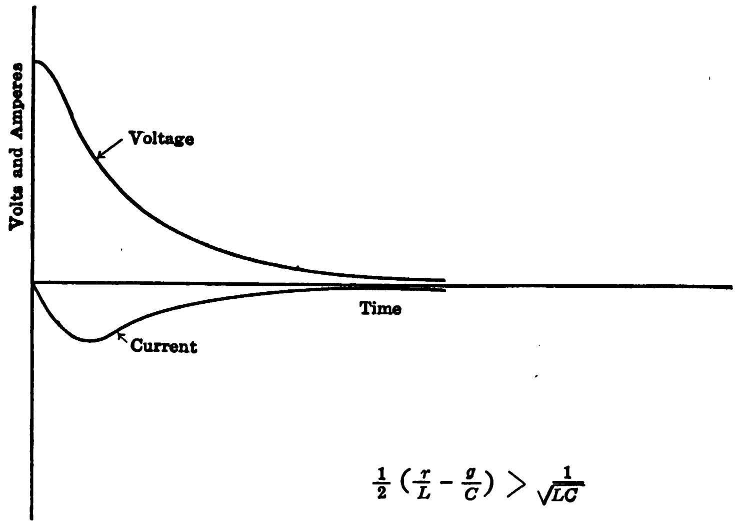


FIG. 121.—Discharge of a condenser through a coil.

Problem 11.—The coil in Problem 10 has a resistance of 5 ohms. The insulation resistance of the condenser is 100,000 ohms. Using the exact relations, equations (29) to (30c), calculate:

(*a*) The attenuation constant of the circuit. (*b*) The distortion constant. (*c*) The frequency constant. (*d*) The natural frequency. (*e*) If the condenser is charged to 1000 volts just before it is connected to the coil, plot to scale for 3 complete cycles the instantaneous values of the voltage and current.

Answer.—(*a*) $u = 17.9$. (*b*) $q = 17.7$. (*c*) $\omega_o = 377$. (*d*) 60 cycles per second. (*e*) One point on curve: $t = \frac{1}{240}$, $v = 43.6$ volts, $i = 17.5$ amperes.

159. Electrostatic Energy.—As noted in the previous article, an electric field in a dielectric represents a definite amount of

stored energy, just as a magnetic field represents a definite amount of stored energy. Since an electric field in a dielectric may be due entirely to electricity at rest, the stored energy of an electric field is usually called "electrostatic" energy. It is analogous in many respects to the potential energy stored in a stretched spring (see Problem 14, p. 25), whereas, as noted in Article 124, magnetic energy (*i.e.*, the stored energy of a magnetic field) is analogous to kinetic energy.

An expression for the amount of electrostatic energy stored in a condenser of constant capacity C when charged to a given potential difference may readily be deduced. Imagine the condenser, initially uncharged, to be connected to any source of electromotive force, such as a battery (see Fig. 115). Let v be the potential drop through the condenser, at any instant, from its positive to its negative plate. The charging current at this instant will then be

$$i = \frac{dq}{dt} = C\frac{dv}{dt}.$$

The electric power input to that portion of the electric circuit included between the two plates of the condenser, viz., to the dielectric between these plates, is then

$$p = vi = Cv\frac{dv}{dt}$$

The energy input to the dielectric in time dt is then

$$dW = pdt = Cvdv$$

Whence the total energy input to the condenser when the potential difference between its plates increases from zero to any value v is the integral of this expression between the limits 0 and v, viz.,

$$W = \frac{1}{2}Cv^2$$

This expression is deduced on the assumption that the dielectric of the condenser is a perfect insulator, and therefore that there is no dissipation of energy within it as heat. The energy transferred to the dielectric must therefore be *stored* in it. Whence the electrostatic energy of a condenser of capacity C charged to a potential difference v is

$$\text{Electrostatic energy} = \frac{1}{2}Cv^2 \qquad (33)$$

Compare with the expression for the magnetic energy of a coil of inductance L carrying a current i, viz.,

$$\text{Magnetic energy} = \frac{1}{2} Li^2$$

(See Article 124.)

When a condenser has a leakage conductance g the total current flowing up to its positive plate and away from its negative plate during charge is $i = gv + C\frac{dv}{dt}$. Whence the total power input to the condenser at any instant is

$$p = vi = gv^2 + Cv\frac{dv}{dt}$$

The energy input to the condenser during the interval of time, say t, required to charge it to the potential difference v is then

$$W = \int_0^t gv^2dt + \frac{1}{2}Cv^2 \qquad (34)$$

The first term in this expression represents the energy which is dissipated as heat in the dielectric of the condenser. The second term, which is the same as the value of W given by equation (33), is as before the electrostatic energy which is stored in the condenser.

The energy which is stored in the dielectric of a condenser comes from the source of electromotive force which charges it. When the plates of a charged condenser are connected by a conductor, the discharge of the condenser is due to the transfer of its electrostatic energy to the wire, where it is dissipated as heat or converted into some other form. When the conductor is a coil of appreciable inductance, some of this energy is converted into magnetic energy. When the resistance of the conductor is less than a certain critical value, a surging back and forth of the energy of the system between the condenser and coil will take place, as shown in the preceding article.

Since a charged condenser is a source of electric energy, *i.e.*, can absorb and give out electric energy, it must be considered as a source of electromotive force (see Article 38). Since it absorbs electric energy when it is being charged, *i.e.*, when the current flows *toward* its positive plate, the direction of its electromotive force is such as to oppose this current. When a condenser is discharging it gives out electric energy, and its electromotive force

is therefore in the direction of the current which flows from its positive plate. The numerical value of this electromotive force in either case is equal to the potential difference v between the plates. In short, the electromotive force of a condenser is always numerically equal to the potential difference between its plates, and acts around the circuit in which the condenser is connected in the direction *through* the condenser from its negative to its positive plate. Compare with the electromotive force of a storage battery.

Since the numerical value of the charge on either plate of a condenser of capacity C charged to a potential difference v is $q = Cv$, equation (33) for the electrostatic energy of a stored energy in the dielectric of a condenser may also be written.

$$W = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}qv \tag{35}$$

It can also be shown, by the same method of analysis as employed in Article 116, that the electrostatic energy per unit volume of a dielectric whose dielectric permeability is constant and equal is k, is

$$w = \frac{kF^2}{8\pi} = \frac{D^2}{8\pi k} = \frac{FD}{8\pi} \tag{36}$$

where F and D are the electric intensity and electric flux density respectively in the given volume.

It is important to note that in applying the above formulas all quantities must be expressed in the same system of units.

Problem 12.—Referring to Problem 3, Article 152, the 1-farad condenser there described as made up of tin foil plates 5 mils thick, and paper 10 mils thick will stand a difference of potential of 2000 volts. (The dielectric strength of treated paper is about 250 volts per mil of thickness.) Such a condenser, allowing for the fact that the dielectric sheets must be of greater area than the conducting plates, will have a volume of about 160,000 cubic feet.

(*a*) How many foot-pounds of energy will be stored in this condenser when charged to 2000 volts? (*b*) To what speed could this energy accelerate a mass weighing 100 pounds, assuming no loss due to friction? (*c*) A fly-wheel weighing 1 ton has a diameter of 6 feet. Its radius of gyration is 2.5 feet. At what speed must it be driven in order to have stored in it this same amount of kinetic energy? (*d*) What will be the foot-pounds of energy per cubic foot stored in the condenser? (*e*) If the voltage impressed on this condenser is a sine-wave voltage having a maximum value of 2000 volts and a frequency of 60 cycles per second, namely, $v = 2000 \sin (377t)$, where $377t$ is in radians, what will be the maximum value of the charging current

5

taken by this condenser? (*f*) What would be the volume of a condenser of this same type sufficient to take a maximum charging current of 1000 amperes? (*g*) What would be its electrostatic capacity? (*h*) What would be its linear dimensions if made in the form of a cube? (*i*) What will be the maximum *rate* at which energy is stored in this smaller condenser, *i.e.*, the maximum power input? (*j*) How many times per second will the condenser be charged and discharged?

Answer.—(*a*) 1,475,000 foot-pounds. (*b*) 975 feet per second. (*c*) 834 revolutions per minute. (*d*) 9.22 foot-pounds per cubic foot. (*e*) 754,000 amperes. (*f*) 212 cubic feet. (*g*) 1330 microfarads. (*h*) 5.96 feet on each edge. (*i*) 1000 kilowatts. (*j*) 120 times per second.

Problem 13.—Two parallel metal plates each 1 foot square are ¼ inch apart, and are separated by air. A difference of potential of 1200 volts is established between the two plates, and the source of this potential difference is then disconnected. The insulation resistance between the two plates is to be assumed infinite, and the lines of electric force between them are to be assumed perpendicular to their surfaces and uniformly distributed.

(*a*) What is the value of the charge acquired by each plate? (*b*) How much energy is stored in the electric field between the plates? (*c*) Will the charge on each plate change in amount if the plates are moved with respect to each other, remaining perfectly insulated? (*d*) Will the difference of potential between the plates change when they are moved with respect to each other, and if so, by how much? (*e*) Will the energy of the electric field increase or decrease if the distance between the plates is increased, and where does this energy come from? (*f*) By how much will the difference of potential between the plates and the energy of the electric field change if the distance between the plates is doubled? (*g*) What is the value of the force required to separate the plates, *i.e.*, what is the force exerted by one plate on the other due to the charges on them? (*h*) If, when the plates are in their original position, a plate of glass 1 foot square and ⅛ inch thick is inserted between and parallel to them, without touching either, will the force exerted on either plate be altered? (*i*) By how much will the potential difference between the plates and the energy of the electric field be changed by inserting the glass? (The specific inductive capacity of the glass is 7.) (*j*) If the plate of glass is held so that a part of it projects outside the space between the metal plates, will there be a force exerted on the glass, and if so, what will be the direction of this force?

Answer.—(*a*) 466 statcoulombs. (*b*) 933 ergs. (*c*) No. (*d*) Yes, since the charge remains constant, whereas the capacity of the condenser decreases. When one plate is moved a distance x parallel to the direction of the lines of force, the number and distribution of the lines of force will not change (provided x is small compared with the diameter of the plates). Hence the electric intensity F will not change, and therefore the increase in the potential difference between the plates is proportional to the displacement x. (*e*) The electrostatic energy increases with an increase in the distance apart of the plates, for the charges remain constant, whereas the capacity of the condenser decreases. This increase in energy results from the mechanical work done by the agent which moves the plate. (*f*) The potential differ-

ence is doubled and the electrostatic energy is doubled. (*g*) 2205 dynes or 0.075 ounce. (*h*) No. (*i*) The potential difference will be decreased by 514 volts and the electrostatic energy will be decreased by 399 ergs. (*j*) There will be a force tending to pull the glass into the space between the plates. The force is therefore in such a direction as to produce a motion which will *decrease* the electrostatic energy of the electric field. (Note an important similarity between electrostatic and magnetic energy, viz., the force exerted on a dielectric by an electric field tends to move the dielectric into such a position as will increase the electric flux through it, just the force exerted on a magnetic substance in a magnetic field tends to move this substance into such a position as will increase the magnetic flux through it.)

160. Forces Exerted by Charged Bodies on One Another.—Experiment shows that every charged body in an electric field is acted upon by a mechanical force, due to the agent which produces this field. In particular, every charged conductor exerts a mechanical force on every other charged conductor in its vicinity, as illustrated in Problem 13 of the last article. Mechanical forces are also in general exerted by an electric field on any dielectric in it when this dielectric has a specific inductive capacity different from that of the surrounding medium, as illustrated in the last problem by the force tending to pull the plate of glass into the field between the charged plates of the condenser.

The value of mechanical force exerted by an electric field (strictly, by the charges which produce this field) on a conductor or dielectric may always be deduced from the general principle that **the component of this force in any direction is equal to the decrease in the electrostatic energy of the field per unit displacement of the given body in this direction,** when all the real charges in the field remaining constant in amount and distribution. This fundamental law is identical with the corresponding law for the mechanical force exerted by a magnetic field (see Article 129), except that the positive sense of the mechanical force due to a magnetic field is in the direction of a displacement corresponding to a *decrease* of magnetic energy (when all the currents in the field remain unaltered), whereas the positive sense of the mechanical force due to an electric field is in the direction of a displacement corresponding to an *increase* of electrostatic energy (when all the real charges in the field remain unaltered).

The principle just stated may be expressed mathematically as follows: Let W be the electrostatic energy of the given electric field, and let dW be the increase in this energy when a given body

(conductor or dielectric) in this field is moved a distance dx in a given direction, all the charges in the field remaining constant in amount and distribution. Then the component, in the direction of dx, of the force exerted on the given body by all the other charged bodies in the field is

$$f_x = -\frac{dW}{dx} \tag{37}$$

This general principle applied to the special case of an electric field due to equally and oppositely charged conductors gives a relatively simple expression for the mutual mechanical force exerted by these conductors on each other. Two such conductors, irrespective of their size, shape or relative position, form a condenser; let C be the capacity of this condenser in c.g.s. electrostatic units. Let Q be the numerical value of the charge on each of these conductors, in c.g.s. electrostatic units. From equation (35), the electrostatic energy of the electric field due to the charges on the two conductors is then

$$W = \frac{1}{2}\frac{Q^2}{C} \qquad \text{ergs}$$

Let one of these conductors be given a linear displacement of dx centimeters in any direction. The corresponding *decrease* in the electrostatic energy W, *per unit displacement* of the given conductor in this direction, is then

$$-\frac{dW}{dx} = -\frac{Q^2}{2}\frac{d}{dx}\left(\frac{1}{C}\right) \qquad \text{ergs per centimeter}$$

Hence the component, in the direction of dx, of the force exerted by one conductor on the other, due to the charges Q and $-Q$ on them, is

$$f_x = -\frac{Q^2}{2}\frac{d}{dx}\left(\frac{1}{C}\right) \qquad \text{dynes} \tag{38}$$

This relation may also be written

$$f_x = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \frac{V^2}{2}\frac{dC}{dx} \qquad \text{dynes} \tag{38a}$$

where V is the potential difference between the two conductors.

Since the capacity of a condenser *decreases* as the distance between its plates *increases*, this force has a positive value in the direction corresponding to a motion of one conductor toward another. This means that the force between two oppositely charged conductors is a force of attraction, which is in accord

with the elementary experimental fact that bodies charged with electricity of opposite signs attract each other; whereas bodies charged with electricity of like sign repel each other.

The formula usually given in elementary text-books for the mechanical force exerted by one charged body on another is

$$f = \frac{qq'}{kx^2} \qquad \text{dynes} \tag{39}$$

where q and q' are the charges on the two bodies in statcoulombs, k is the specific inductive capacity of the surrounding medium, and r is the distance between the two bodies in centimeters. This formula, however, is applicable only (1) when the two bodies are relatively small compared to their distance apart, and (2) when the surrounding space has in it no dielectric bodies whose specific inductive capacity is different from that of the medium in contact with the charged bodies. This relation is entirely consistent, under the conditions just specified, with the general principle above stated. However, equation (39) is of little value for practical calculations, since the first of the two conditions is practically never satisfied, and the second condition is also frequently not fulfilled. Compare with the corresponding expression for the mechanical force between magnetic poles, Article 134.

The practical case of most frequent occurrence is that of two equally and oppositely charged conductors, to which case equation (38) is always applicable, irrespective of the shape, size or relative position of the two conductors, and also irrespective of the uniformity of the dielectric between them. The effect of the shape, size and relative position of the two conductors, and the nature of the dielectric, are all taken into account in the expression for the capacity C.

Problem 14.—From equation (7) the electrostatic capacity per centimeter length of a two-wire transmission line is, in c.g.s. electrostatic units,

$$C = \frac{K}{4 \log_\epsilon \frac{D}{r}}$$

where K is the specific inductive capacity of the surrounding medium, D is the distance between the axes of the two wires and r is the radius of each wire. Prove that the force of attraction between the two wires, due to the charges of on them of q and $-q$ *coulombs per foot*, is

$$f_x = 5.21 \times 10^{11} \frac{lq^2}{KD} \qquad \text{pounds} \tag{39}$$

where l is the length of each wire in feet, and D is the distance between the centers of the two wires in inches. Compare with equation (5), in Problem 2, Article 130.

Problem 15.—A certain overhead, two-wire transmission line consists of stranded aluminum wires having a cross-section of 500,000 circular mils, and spaced 36 inches between centers. The difference of potential between the two wires is 30,000 volts, and the current in each wire is 200 amperes. The wires are supported on insulators spaced 100 feet apart.

(*a*) What is the force exerted on each insulator due to the current in the wires? (*b*) What is the charge per foot on each wire? (*c*) What is the force exerted on each insulator due to the charges on the two wires? (*d*) What is the resultant force on each insulator due to the combined effect of the current and charges?

Answer.—(*a*) Repulsion of 0.06 pound = 0.96 ounce. (*b*) 5.67×10^{-8} coulombs per foot. (*c*) Attraction of 0.00466 pound = 0.0746 ounce = 31.6 grains. (*d*) Repulsion of 0.89 ounce. (NOTE.—The relative values here found for the repulsion between two wires due to the current in them and the attraction due to the charges on them are typical of the usual conditions which arise in practice, *i.e.*, the mechanical forces due to magnetic effects are much larger than those due to electrostatic effects.)

161. Parallel-plate Electrometer. Electrostatic Voltmeter.—The attraction between two conductors when a difference of electric potential is established between is utilized, in certain

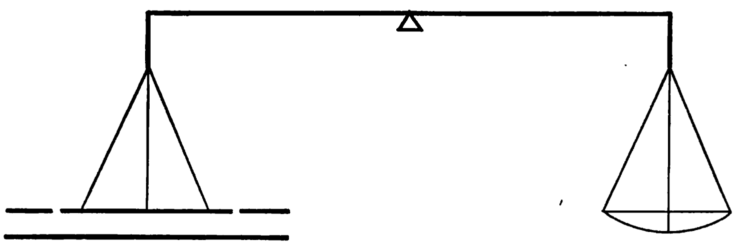


FIG. 122.—Parallel-plate electrometer.

types of measuring instruments, for determining the value of the potential difference impressed. Such instruments are called "electrostatic voltmeters." The simplest form of electrostatic voltmeter consists of two parallel plates, mounted in horizontal planes, with the upper plate hung from the arm of a balance, as indicated in Fig. 122. This type of electrostatic voltmeter is called a "parallel-plate electrometer."

The upper plate usually consists of a disc surrounded by a flat ring, called a "guard ring," separated from the disc by a narrow air-gap, but connected to it electrically by a fine wire. The guard ring is fixed relative to the lower plate, and is insulated

from it. Only the disc is supported from the arm of the balance and is free to move. The object of the guard ring is to insure a uniform distribution of the charge on the movable disc, so that the lines of force which leave it are uniformly distributed in the air space between the plates.

Let V be the difference of potential established between the plates, in c.g.s. electrostatic units; let x be the distance between the disc and the lower plate, in centimeters; let S be the area of the disc in centimeters; and let f be the force, in dynes, exerted on the disc by the lower plate, when the planes of the under surfaces of the disc and guard ring coincide. The electric intensity in the air between the disc and lower plate is then

$$F = \frac{V}{x}$$

The volume of the air space between the disc and lower plate is Sx. Hence, from equation (36), the electrostatic energy in this space is

$$W = \frac{V^2}{8\pi x^2} \times (Sx) = \frac{V^2 S}{8\pi x}$$

Consequently, from equation (37), the force exerted by the lower plate on the disc is

$$f = \frac{V^2 S}{8\pi x^2}$$

Whence the value of the potential difference between the plates is

$$V = x\sqrt{\frac{8\pi f}{S}} \qquad \text{statvolts} \qquad (40)$$

Consequently, by measuring the distance x between the plates, the area S of the movable disc, and the force f, the value of the potential difference V may be found directly in c.g.s. electrostatic units. For this reason, this type of electrometer is sometimes called an "absolute" electrometer.

For a description and theory of other types of electrostatic voltmeters see any text-book on electrical measurements.

Problem 16.—A source of high, constant voltage is connected to the plates of a parallel plate electrometer, and it is found that 5 grams is necessary to balance the pull on the disc when the plates are 1 centimeter apart. The diameter of the movable disc is 25 centimeters. What is the value of the impressed voltage?

Answer.—15.8 statvolts = 4740 volts.

XIV

SINE-WAVE ALTERNATING CURRENTS

162. Introduction.—As shown in Article 128, an alternating electromotive force, *i.e.*, an electromotive force which passes over and over again through a complete cycle of positive and negative values, may be produced by revolving a coil in a constant magnetic field. A simple form of generator for producing an alternating electromotive force is illustrated in Fig. 93. As already noted in Article 123, a generator which produces an alternating electromotive force is called an alternating-current generator, or, more briefly, an alternator.

When an alternating electromotive force is impressed across the terminals of an electric circuit of any kind, an alternating current of the same frequency is established in this circuit.[1] When the strength of the current at any instant in the given circuit is i, and the potential drop through this circuit in the direction of the current is v, the input of electric energy to the circuit in an infinitesimal interval of time dt measured from this instant is equal to $vidt$. The rate at which energy is transferred to this circuit at this instant, or the instantaneous *power input*, is $p = vi$

Although the average values of an alternating current and an alternating voltage over a complete period are both zero, the average value of the power input (or output, when the potential drop is in the opposite direction to the current) is *not zero*, except in certain special cases (see Article 167). Hence an alternating current may be used to transmit electric energy, just as a direct current is used for this purpose.

The chief advantages of alternating over direct current are the following:

[1] The current established for the first second or two after the alternating electromotive force is impressed is not strictly an *alternating* current, since its amplitude builds up gradually to a constant maximum, just as the value of the current established in a circuit by a continuous electromotive force builds up gradually from zero to a constant maximum, depending upon the resistance, inductance and capacity. See the article on *Transient Electric Phenomena* in Pender's *Handbook for Electrical Engineers*.

1. Alternating-current generators are much less expensive for the same power output, since only slip rings, instead of an elaborate commutator, are required for making connection to the external circuit.

2. Certain forms of alternating-current motors, particularly the induction motor, are cheaper to build and require less care in operation than a direct-current motor of the same power output.

3. The greatest advantage of alternating currents is that, by means of a transformer (see Article 127), power may be readily transferred from a circuit in which the current is small and the voltage high to a circuit in which the voltage is low and the current large. Since the power lost in a transmission line depends upon the *square* of the current, it is obvious that for economical transmission the current should be kept small, and therefore the voltage high. On the other hand, a high voltage is dangerous, particularly inside of buildings. Hence electric power is usually generated by an alternator of comparatively low voltage, "stepped up" by means of a transformer to a high voltage for transmission, and then "stepped down" by means of another transformer to a comparatively low voltage for local distribution and use.

In the discussion of the generation, transmission and utilization of alternating-current power various terms, in addition to those already defined in the previous chapters of this book, are commonly employed. The fundamental principles involved, however, are simply those which have already been discussed. The "theory" of alternating currents is nothing more than the application of these general principles to electromotive forces and currents which vary cyclicly with time.

163. General Definitions.—The following definitions are stated in terms of current, but are equally applicable to electromotive forces, potential differences, magnetic fluxes, or any other quantity which varies with time.

Alternating Current.—An alternating current is a current which varies continuously with time, from a constant maximum value in one direction to a constant maximum value in the opposite direction, and back again to the same maximum in the first direction, and whose average value for a complete cycle is zero.

Wave Form.—In Fig. 123 the successive instantaneous values of an alternating current are plotted as ordinates against time as the abscissa. Such a curve is called the "wave form," or "wave shape," of the current. In the figure the positive and negative portions of the current wave are shown unsymmetrical; such a

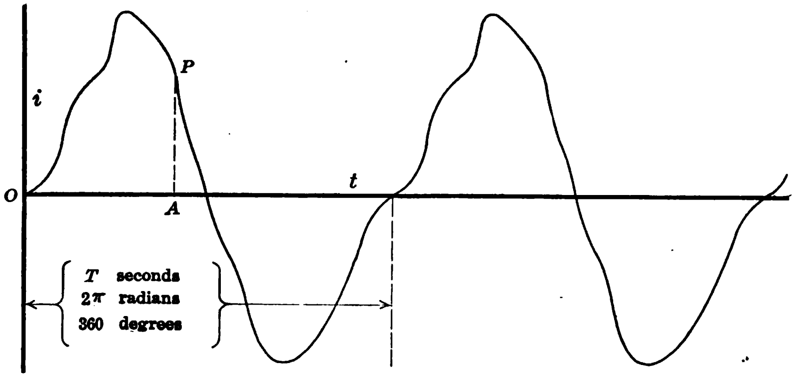


FIG. 123.—Wave-form.

non-symmetrical current or voltage wave is physically possible, but the current and voltage waves developed in ordinary electric machines are usually perfectly symmetrical, *i.e.*, the positive and negative portions of the wave are exactly alike. A wave

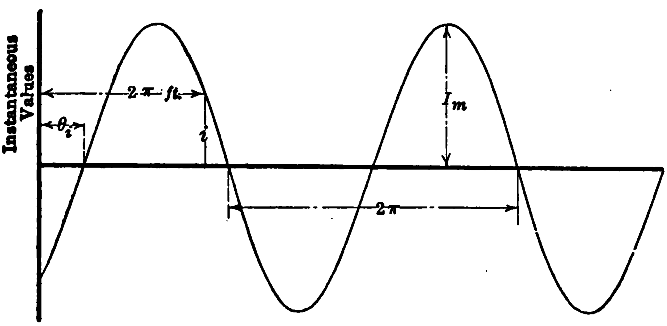


FIG. 124.—Sine-wave.

form which is a pure sine curve, or cosine curve, is called a "sine wave;" see Fig. 124.

Phase.—The abscissa of any point on a wave form is called the "phase" of the ordinate at this point, referred to the origin of the system of co-ordinates employed. For example, the phase of the ordinate $\overline{AP}$ in Fig. 123 is the distance $\overline{OA}$.

When the scale of abscissas is so chosen that the distance corresponding to a complete cycle is 2π scale divisions (viz., when the distance corresponding to the time T is equal to 2π), the phase of any ordinate may be expressed as so many radians. Similarly, when the scale of abscissas is so chosen that the distance corresponding to a complete cycle is 360 scale divisions, the phase of any ordinate is expressed in degrees. For example, the phase of the ordinate $\overset{\frown}{AP}$ in Fig. 123 is 2.4 radians or 137 degrees.

Period.—The period of an alternating current is the time taken for the current to pass through a complete cycle of positive and negative values; *i.e.*, the period is equal to the time T indicated in Fig. 123. A complete period corresponds to 2π radians or 360 degrees.

Frequency.—The frequency, or number of cycles per second, is the number of periods per second.

Alternations.—The number of alternations per minute is the total number of times per minute that the current changes in direction, from positive to negative and from negative to positive. In engineering practice the number of cycles is usually referred to the second as the unit of time, and the number of alternations is referred to the minute as the unit of time

Let T be the period and f the frequency, or number of cycles per second; then

$$f = \frac{1}{T} \tag{1}$$

$$\text{Alternations per minute} = 120f = \frac{120}{T} \tag{2}$$

The electromotive force induced in the armature winding of an alternator reverses in direction every time the armature rotates through the angle corresponding to the distance between successive field poles. Hence, calling N the number of revolutions per minute and p the number of field poles, the number of alternations of the electromotive of the alternator is

$$\text{Alternations per minute} = Np$$

The frequency of this electromotive force is therefore

$$f = \frac{Np}{120} \tag{3}$$

Instantaneous Value.—The instantaneous value of an alternating current is the value of the current at any instant. The

ordinates of the wave form of any alternating quantity give its instantaneous values. The phase of this quantity at any instant is the phase of the corresponding ordinate, and depends, of course, upon the instant chosen as the "zero" of time.

Maximum Value.—The maximum value of an alternating current is the greatest instantaneous value during any cycle. The maximum value of any alternating quantity is the maximum ordinate in its wave form. The maximum value of an alternating quantity is also called its "crest value."

Average Value.—The average value of an alternating current over a *complete* cycle is zero. The term "average value" is used, however, to designate the numerical value of the average of its instantaneous values between successive zero values. This is equal to the average ordinate of half its wave form.

R.M.S., or Effective, Value.—The r.m.s., or effective, value of an alternating current is the square root of the mean of the squares of its instantaneous values over a complete period. The abbreviation "r.m.s." stands for "root-mean-square." In England the term "virtual value" is used to designate this same quantity. The r.m.s. value of any alternating quantity may be found graphically by plotting a curve whose ordinates are the squares of the ordinates of the wave form of this quantity, finding the average ordinate of this new curve, and taking the square root of this average ordinate.

The definition of r.m.s. value of any varying quantity may be expressed mathematically as follows: Let i be the value of this quantity at any instant, and let T be the period of its variation (*i.e.*, the time taken for it to pass through a complete cycle of positive and negative values). Then the r.m.s. value of this quantity is

$$\text{R.m.s. value} = \sqrt{\frac{1}{T}\int_o^T i^2 dt} \tag{4}$$

It is the r.m.s. value of an alternating current which is indicated by an alternating current ammeter. Similarly, it is the r.m.s. value of an alternating voltage which is indicated by an alternating-current voltmeter. The r.m.s. value of an alternating current may therefore be thought of as its "ammeter" value, and the r.m.s. value of an alternating voltage may be thought of as its "voltmeter" value.

Any type of instrument in which the torque which produces the deflection of the movable element depends on the *square* of the current through it, or on the *square* of the voltage impressed on it, may be used as an alternating-current ammeter or voltmeter respectively.

Consider, for example, an electrodynamometer type of ammeter. The stationary and movable coil being in series, the current in each of the two coils is the same. Let the value of this current at any instant be i. From Article 131, the torque exerted on the movable coil by the fixed coil is, at this instant, ki^2, where k is a constant whose value depends upon the mutual inductance of the two coils. When the current i is a direct current of value I, then this torque is constant, and has the value kI^2. When the current is an alternating current, the torque changes at each instant. However, when the natural period of mechanical oscillation of the movable coil is small compared with the period of the current, the deflection of this coil will be determined by the *average* value of the torque acting on it, *i.e.*, by the average value of ki^2 over a complete period. The deflection of the movable coil will, therefore, be the same as would be produced by a direct current of such a value I that kI^2 = average of ki^2, or

$$I = \sqrt{\text{average of } i^2}$$
$$= \sqrt{\frac{1}{T}\int_0^T i^2 dt}$$

This is the r.m.s. value as given by equation (4).

Hence an electrodynamometer type of ammeter, calculated to read correctly the value of a direct current, will likewise give the r.m.s. value of an alternating current. In an exactly similar manner it may be shown that a current balance (Article 33) may be used to measure the r.m.s. value of an alternating current. An ammeter whose movable element is a soft iron core or vane may also be used as an alternating-current ammeter. What has just been said in regard to ammeters also holds with respect to voltmeters.

An ammeter or voltmeter whose stationary or movable element is a *permanent* magnet cannot be so used, for in such an instrument the torque is proportional to the average value of the *first power* of the current. The average value of the torque exerted

on the movable element of such an instrument when an alternating current is sent through it is therefore zero. The ordinary D'Arsonval type of meter is therefore not suitable for alternating current measurements.

Form Factor.—By the form factor of a wave is meant the ratio of its r.m.s. ordinate to the average ordinate of its positive (or negative) half wave. That is, calling I the r.m.s. ordinate and $I_{aver.}$ the average ordinate,

$$\text{Form factor} = \frac{I}{I_{aver.}} \tag{5}$$

Crest Factor.—By the crest factor of a wave is meant the ratio of its maximum ordinate to its r.m.s. ordinate. That is, calling I_m the maximum ordinate, and I the r.m.s. ordinate,

$$\text{Crest factor} = \frac{I_m}{I} \tag{6}$$

This factor is also called the "amplitude factor," or "peak factor."

Problem 1.—An alternator which has 18 poles is driven at a speed of 400 revolutions per minute.

(*a*) What is the number of alternations of its electromotive force? (*b*) What is the frequency of this electromotive force? (*c*) What will be the frequency of the current established by it in a circuit connected to its terminals? (*d*) At what speed would this alternator have to be driven to produce a 25-cycle current (*i.e.*, a current whose frequency is 25 cycles per second)?

Answer.—(*a*) 7200 alternations per minute. (*b*) 60 cycles per second. (*c*) The same, namely, 60 cycles per second. (*d*) 167 revolutions per minute.

Problem 2.—(*a*) Replot the wave form shown in Fig. 123, choosing the zero of time in such a manner that the phase of the point P will be $+ 30$ degrees. (*b*) What will be the phase of the first zero value to the right of this origin? (*c*) Of the first zero value to the left of the origin? (*d*) What will be the phase of the positive maximum nearest the origin?

Answer.—(*b*) 60 degrees. (*c*) $-$ 120 degrees. (*d*) $-$ 25 degrees.

Problem 3.—A symmetrical wave each half of which is a rectangle may be called a rectangular wave. A symmetrical wave, each half of which is an isosceles triangle may be called a triangular wave. A symmetrical wave, each half of which is a circle, may be called a semicircular wave. Calling I_m the maximum ordinate of each of these waves, prove the relations given in the following table:

	Rectangle	Triangle	Semicircle
Average value....................	I_m	$\frac{1}{2} I_m$	$\frac{\pi}{4} I_m$
R.m.s. value.....................	I_m	$\frac{1}{\sqrt{3}} I_m$	$\sqrt{\frac{2}{3}} I_m$
Form factor......................	1	1.15	1.04
Crest factor.....................	1	1.73	1.22

(NOTE.—The equation of each half of the semicircular wave is $y^2 = I^2_m - x^2$, and the value of x corresponding to half a period is equal to $2I_m$. The equation of the first quarter of the triangular wave is $y = \frac{4I_m}{T} x$, where T is the period of this wave, and the value of x corresponding to quarter of a period is $\frac{T}{4}$.)

164. Sine-wave Quantities.—A sine-wave quantity is an alternating quantity whose wave form is a sine wave, viz., whose wave form is a curve (Fig. 124) whose equation is of the form

$$i = I_m \sin(2\pi ft - \theta_i) \tag{7}$$

where the angle $(2\pi ft - \theta)$ is in *radians*, and

I_m = the maximum value of the given quantity,

f = the frequency, or number of times per second that the given quantity passes through a complete cycle of positive and negative values,

t = time in seconds, measured from any arbitrarily chosen instant, or "zero" of time,

θ_i = the phase of the first *ascending* zero value of the given quantity (see Fig. 124). This angle is called the "phase angle" of the given quantity referred to the given zero of time.

A sine-wave quantity is also called a "sinusoidal" quantity. Still another name for a sine-wave quantity is a "simple harmonic" quantity. A quantity whose wave form is not a sine wave is called a "non-sinusoidal" quantity.

In practice, most alternating currents and electromotive forces have wave forms which are approximately sine waves. This is particularly fortunate, for the properties of sine-wave currents and electromotive forces are relatively simple. Moreover, as will be shown in Chapter XVIII, any alternating quantity, no matter how irregular its wave form, may be resolved into a sum of

sine-wave components whose frequencies are integer multiples of the frequency of the given quantity. Consequently, the theory of alternating currents is primarily a study of the properties of sine-wave quantities.

The quantity $2\pi f$ which enters into the mathematical expression for a sine-wave current may be called the "frequency constant" of the current. The frequency constant is usually represented by the symbol ω, viz.,

$$\omega = 2\pi f \tag{8}$$

Other names used for this same quantity are "angular frequency" and "periodicity." The symbol "p" is used by some authors to designate it.

In terms of the frequency constant ω, the equation for a sine-wave quantity is of the form

$$i = I_m \sin (\omega t - \theta_i) \tag{9}$$

This is the form usually employed.

The frequencies commonly employed in this country for power and lighting purposes are 25 and 60 cycles per second respectively, for which the corresponding values of ω are

For 25 cycles per second:	$\omega = 157$
For 60 cycles per second:	$\omega = 377$

As already noted, the angle $2\pi ft$ in equation (7), and the corresponding angle ωt in equation (9), are in *radians*, not degrees. When it is desired to use degrees instead of radians, the number 360 must be substituted for 2π.

In plotting a sine-wave quantity, it is usually more convenient, instead of taking time t as abscissas, to take the abscissas equal to (ωt) or $(360\,ft)$, accordingly as it is desired to express angles in radians or in degrees. For plotting curves to scale the degree is the most convenient unit for the angle. However, in any problem involving a differentiation or integration of a given sine-wave quantity, the angles are ususally expressed in radians, as otherwise confusing conversion factors come into the formulas.

The value of the phase angle θ_i in the above expression $i = I_m \sin(\omega t - \theta_i)$ for a sine-wave current depends solely upon the instant chosen as the "zero," or "origin," of time. For example, when the zero of time is chosen as the instant at which

the current in a given circuit is zero and is *increasing* (see Fig. 125), the equation of this current is

$$i = I_m \sin \omega t \tag{9a}$$

When the zero of time is chosen as the instant at which a given sine-wave quantity has its maximum positive value, the phase angle of this particular quantity is $-\frac{\pi}{2}$ radians or $-$ 90 degrees. For example, when the zero of time is chosen as the instant at which the current in a given circuit has its maximum positive value (see Fig. 127), the equation of this current is

$$i = I_m \sin \left(\omega t + \frac{\pi}{2}\right) \tag{9b}$$

or

$$i = I_m \cos \omega t \tag{9c}$$

When the zero of time is chosen as the instant at which a given sine-wave quantity is zero, and is *decreasing,* its phase angle is π radians or 180 degrees. For example, when the zero of time is chosen as the instant at which the current in a given circuit is zero and is *decreasing* (see Fig. 126), the equation of this current is

$$i = I_m \sin (\omega t - \pi) \tag{9d}$$

or

$$i = -I_m \sin \omega t \tag{9e}$$

165. R.M.S. Value of a Sine-wave Quantity.—The substitution of equation (7) in the general definition of r.m.s. value given by equation (4) gives for the r.m.s. value of a sine-wave quantity the expression

$$I = \sqrt{\frac{1}{T}\int_0^T I^2{}_m \sin^2 (2\pi ft - \theta_i)dt}$$

Noting that in general $\sin^2 x = \frac{1 - \cos 2x}{2}$, where x is any angle, and that the integral of $\cos 2x$ over a complete cycle of x (*i.e.* from $x = 0$ to $x = 2\pi$) is zero, it follows that,

$$\int_0^T \sin^2 (2\pi ft - \theta_i)dt = \int_0^T \frac{dt}{2} = \frac{T}{2}$$

Whence the r.m.s. value of a sine-wave quantity whose maximum value is I_m is

$$I = \frac{I_m}{\sqrt{2}} \tag{10}$$

That is, **the r.m.s. value of any sine-wave quantity is equal** *to* **its maximum value divided by $\sqrt{2}$.**

It should be carefully noted that this relation applies only to SINE-WAVE *quantities.* When the wave form of an alternating quantity is not a sine curve the ratio of its maximum value to its r.m.s. value is NOT equal to $\sqrt{2}$ (see Problem 3, Article 163).

Problem 4.—A sine-wave alternating current has an r.m.s. value of 100 amperes and passes through 5 complete cycles of values in $\frac{1}{5}$ second. (*a*) What is the maximum value of this current? (*b*) What is the period of this current? (*c*) What is its frequency? (*d*) What is the value of its frequency constant? (*e*) Write the equation of this current referred to the instant at which the current is zero and increasing? (*f*) Referred to the instant at which the current has its maximum positive value. (*g*) What is the equation of this current when the zero of time is taken as the instant at which the current is 50 amperes and is increasing? (*h*) What is the phase angle of this current referred to this zero of time? (*i*) Plot this last equation to scale for a complete cycle, taking 1 inch on the scale of abscissas equal to 100 degrees, and 1 inch on ordinate scale equal to 40 amperes.

Answer.—(*a*) 141 amperes. (*b*) $\frac{1}{25}$ second. (*c*) 25 cycles per second. (*d*) 157. (*e*) $i = 141 \sin (157t)$. (*f*) $i = 141 \cos (157t)$. (*g*) $i = 141 \sin (157t + 0.363)$ when the angle is in radians, or $i = 141 \sin (9000t + 20.7)$ when the angle is in degrees. (*h*) -20.7 degrees or -0.363 radians.

166. Difference in Phase.—When two sine-wave quantities of the same frequency pass through their zero values at different instants of time, they are said to differ in phase. The relative positions of the wave forms of a current and voltage which differ in phase by an angle θ is shown in Fig. 128. From the figure it is evident that this phase difference is equal to the difference between the individual phase angles of the current and voltage referred to any arbitrarily chosen zero of time.

The zero of time may always be chosen as the instant at which one, *but only one,* of two or more alternating quantities passes through its zero value in the ascending direction. This particular quantity may then be referred to as the quantity of reference, and its equation is then of the form of equation (9*a*). The difference in phase between this and any other quantity referred to this same zero of time is then numerically equal to the phase angle of this latter quantity.

When two sine-wave quantities pass through their zero values at the same instant, and both are increasing in the positive direction, as shown in Fig. 125, their phase angles are equal, and these two quantities are said to be "in-phase."

When the two quantities pass through their zero values at the same instant, but one is increasing and the other decreasing, as shown in Fig. 126, they differ in phase by π radians, or 180 degrees. They are then said to be "in opposition."

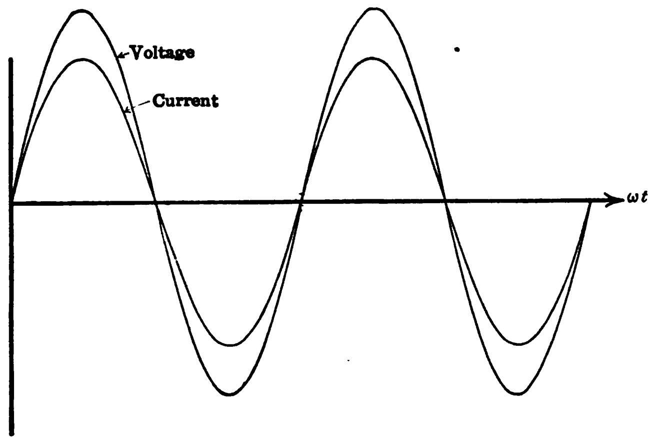


FIG. 125.—Current and voltage in phase.

When one of the two quantities passes through its zero value at the same instant that the other quantity passes through its maximum value, as shown in Fig. 127, they differ in phase by $\frac{\pi}{2}$ radians, or 90 degrees. They are then said to be in "quadrature."

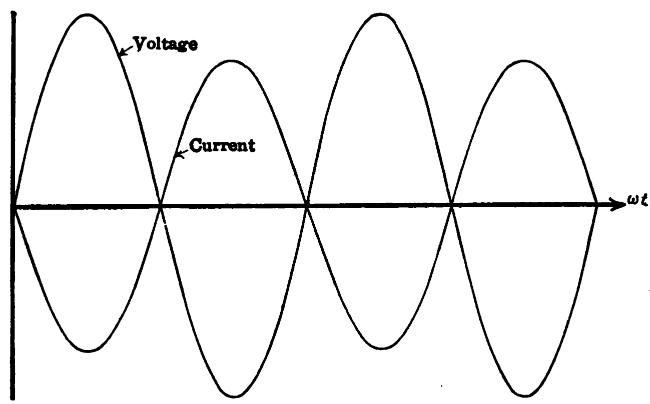


FIG. 126.—Current and voltage in opposition.

Of two sine-wave quantities which differ in phase, that quantity which reaches its positive maximum first, when time is measured from any given instant, is said to "lead" the other

by the angle numerically equal to the algebraic difference of their phase angles. Conversely, the quantity which reaches its first positive maximum last is said to "lag" the other quantity.

For example, in Fig. 127 the current leads the voltage by $\frac{\pi}{2}$

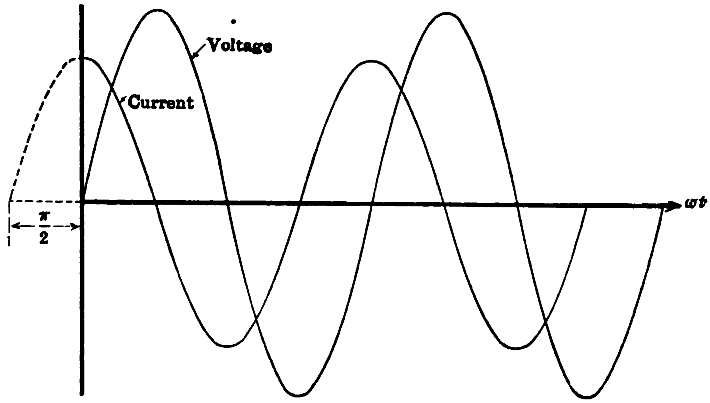


FIG. 127.—Current and voltage in quadrature.

radians or 90 degrees, and the voltage lags the current by this same angle. In Fig. 128 the current lags the voltage by the angle θ, equal to the difference between the phase angles of the current and voltage, and the voltage leads the current by this same angle.

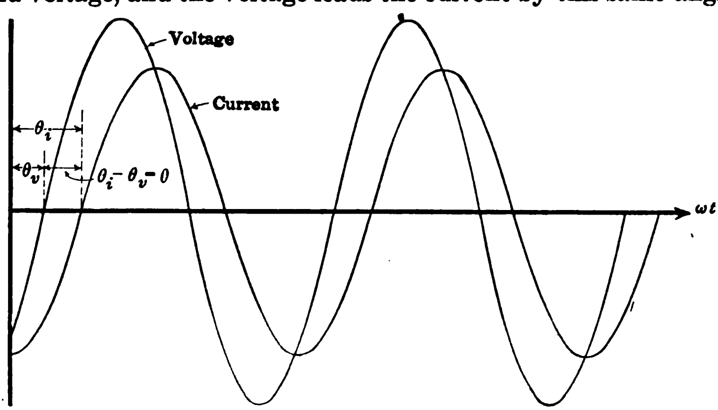


FIG. 128.—Current lagging voltage by the angle θ.

From these two figures it is evident that the curve which is shifted to the *right* with respect to the other is the curve which *lags*. Or, looked at from another point of view, the quantity which has the *algebraically larger* phase angle (as defined by

equation (7)) lags. For example, when the origin of time is so chosen that the voltage has a negative phase angle of 20 degrees (see definition of phase angle, p. 79) and the current a positive phase angle of 15 degrees, the current lags the voltage by $15 - (-20) = 35$ degrees.

Problem 5.—A sine-wave current whose r.m.s. value is 50 amperes is produced in a circuit by impressing on the terminals of this circuit a sine-wave voltage whose r.m.s. value is 300 volts and whose frequency is 60 cycles per second. At the instant that the current is zero and is increasing, the voltage drop through the circuit in the direction of the current is -100 volts and is decreasing in numerical value.

(*a*) What is the difference in phase between the current and the volt drop through the circuit in the direction of the current? (*b*) Does the current lead or lag the voltage? (*c*) Plot this voltage drop and current for one complete cycle of the latter, taking 1 inch on the scale of abscissas equal to 100 degrees, and 1 inch on the ordinate scale equal to 100 volts for the voltage curve and 20 amperes for the current curve.

Answer.—(*a*) 13.7 degrees. (*b*) The current leads the voltage drop.

Problem 6.—A sine-wave current whose r.m.s. value is 10 amperes, and whose frequency is 25 cycles per second, flows through a non-inductive resistance of 5 ohms.

(*a*) Write the equation for this current, taking as the origin of time the instant at which the current is zero and is increasing. (*b*) Write the equation for the voltage drop through this resistance, in the direction of the current, referred to this same origin of time. (*c*) What is the difference in phase between the current and this voltage drop? (*d*) What is the r.m.s. value of this voltage drop? (*e*) How will an increase in frequency affect the r.m.s. value of the voltage drop?

Answer.—(*a*) $i = 14.1 \sin (157t)$. (*b*) $v = 70.7 \sin (157t)$. (*c*) The current and voltage drop are in phase. (*d*) 50 volts. (*e*) A change in frequency will not affect the voltage drop, provided the resistance is absolutely non-inductive.

Problem 7.—A sine-wave current whose r.m.s. value is 10 amperes, and whose frequency is 25 cycles per second, is established in a coil whose self-inductance is 30 millihenries. The resistance of the coil is negligibly small.

(*a*) Write the equation for this current, taking as the origin of time the instant at which the current is zero and is increasing. (*b*) Write the equation for the voltage drop through the coil, in the direction of the current, referred to this same origin of time. (*c*) What is the difference in phase between the current and this voltage drop? (*d*) What is the r.m.s. value of this voltage drop? (*e*) How will an increase in the frequency affect the r.m.s. value of the voltage drop?

Answer.—(*a*) $i = 14.1 \sin (157t)$. (*b*) $v = 66.6 \cos (157t)$. (*c*) The current lags the voltage drop by $\frac{\pi}{2}$ radians or 90 degrees. (*d*) 47.1 volts. (*e*) The r.m.s. value of the voltage drop is directly proportional to the frequency.

Problem 8.—A sine-wave voltage whose r.m.s. value is 1000 volts, and whose frequency is 60 cycles per second, is impressed across a condenser whose capacity is 2 microfarads and whose leakage conductance is negligibly small.

(*a*) Write the equation for this voltage, taking as the origin of time the instant at which the voltage drop through the condenser from one plate to the other, say from the A plate to the B plate, is zero and is increasing. (*b*) Write the equation for the charging current of this condenser, referred to this same origin of time, calling this current positive when it enters the A plate. (*c*) What is the difference in phase between this current and the voltage drop through the condenser? (*d*) What is the r.m.s. value of the charging current? (*e*) How will an increase in the frequency of the impressed voltage affect the r.m.s. value of the charging current?

Answer.—(*a*) $v = 1414 \sin (377t)$. (*b*) $i = 1.067 \cos (377t)$. (*c*) The charging current leads the voltage by $\frac{\pi}{2}$ radians or 90 degrees. (*d*) 0.754 amperes. (*e*) The r.m.s. value of the charging current is directly proportional to the frequency.

167. Power Corresponding to a Sine-wave Current and a Sine-wave Voltage.—The electric power input at any instant

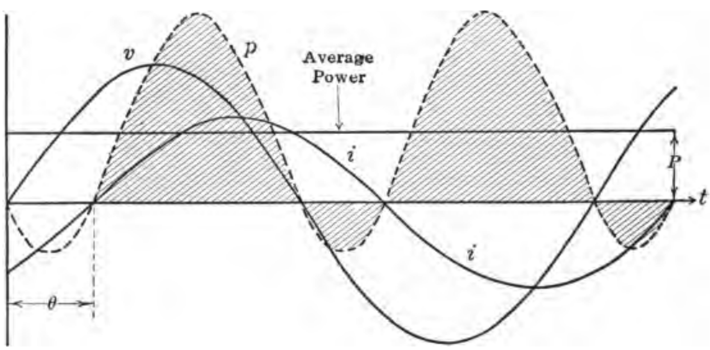


FIG 129.—Power corresponding to sine-wave current and voltage.

to a circuit in which the current has the value i at this particular instant, and in which the drop of potential in the direction of the current at this particular instant is v, is

$$p = vi \tag{11}$$

In Fig. 129 are plotted a sine-wave voltage and a sine-wave current of the same frequency, differing in phase from the voltage by the angle θ. The dotted curve, whose ordinates are equal to the product of the corresponding ordinates of the voltage and current curves, gives the power input to the circuit at each instant during the cycle of their variation. As shown in the figure,

the power input curve is also a sine curve, but of *double* the frequency of the current and voltage curves, and displaced vertically with respect to these curves.

When the current and voltage differ in phase, as shown in the figure, there is an actual *input* of electric energy to the circuit during two intervals in each period, and an actual power *output* of electric energy during the rest of the period. The energy input and output are respectively equal to the shaded areas above and below the time axis. The *resultant energy* input during each cycle is therefore the difference between the two shaded areas above the time axis and the two shaded areas below this axis.

The resultant energy input per cycle divided by the time of one cycle is equal to the average *rate* at which energy is supplied to the circuit, *i.e.*, to the *average power input*, which will be designated by the symbol P. This average power input is equal to the algebraic mean ordinate of the power curve, which in turn is equal to the vertical distance of the axis of symmetry of the power curve above the time axis.

The facts just stated may be readily verified by a comparatively simple mathematical analysis, which will also show that the value of the average power corresponding to a sine-wave voltage of r.m.s. value V and a sine-wave current of r.m.s. value I, when both have the same frequency, is

$$P = VI \cos \theta$$

where θ is the difference in phase between the current and the voltage drop in the direction of the current.

To prove this, choose as the origin of time the instant at which the voltage is zero and increasing. Let θ be the angle by which the current lags the voltage, and let ω be their frequency constant. The equations of the voltage and current may then be written

$$v = \sqrt{2}\ V \sin \omega t$$
$$i = \sqrt{2}\, I \sin (\omega t - \theta)$$

where V and I are the r.m.s. values of these two quantities. The power input at any instant is then

$$p = vi = 2VI \sin (\omega t) \sin (\omega t - \theta) \qquad (12)$$

This last expression may be simplified by making use of the general trigonometric relations that

$$\cos (x - y) = \cos x \cos y + \sin x \sin y$$
$$\cos (x + y) = \cos x \cos y - \sin x \sin y$$

from which by subtraction follows the relation

$$\sin x \sin y = \frac{1}{2} \cos (x - y) - \frac{1}{2} \cos (x + y) \tag{13}$$

The student should memorize this particular relation, for it will be found extremely useful in the analysis of various problems involving the product of two sine functions of the same frequency.

Applying this relation to equation (12), there results

$$p = VI \cos \theta - VI \cos (2\omega t - \theta) \tag{14}$$

Since $\omega = 2\pi f$, where f is the frequency of the current and voltage, it follows that the frequency of the cosine term in this expression is *twice* the frequency of the current and voltage. Consequently a complete cycle of the current or voltage corresponds to two complete cycles of this cosine term. Hence the average value of this cosine term for a complete cycle of the voltage or current is equal to zero.

Consequently, the average value of the power input for a complete cycle is equal to the average value of the term $VI \cos \theta$. But since the factors in this term are all constants, the average value of the power input is equal to this term, viz.,

$$P = VI \cos \theta \tag{15}$$

The student should note carefully that this expression for the average power of an alternating current and voltage *holds only when these quantities are sine-wave quantities* (see Article 211).

In the deduction of equation (15) the angle θ was taken as the angle by which the current lags the voltage. However, since an angle of lead is equivalent to an equal *negative* angle of lag, and since the cosine of a negative angle is equal to the cosine of an equal positive angle, it follows that equation (15) gives the average power input irrespective of whether the current lags the voltage by the angle θ or leads the voltage by this angle.

When the angle θ by which the current lags or leads the voltage drop in the direction of this current is greater than 90 degrees, the cosine of this angle is negative. From equation (15) it then follows that the average power input to the circuit is

negative. This means that there is an actual power *output* from this circuit. Under these conditions it is usually more convenient to take for the phase angle θ the angle by which the current lags or leads the voltage *rise* in the direction of the current. The phase angle θ is then less than 90 degrees, and equation (15) then gives a positive value for P equal to the power *output* of the circuit.

The average power input to (or output from) a circuit is indicated directly by a wattmeter connected to this circuit in the usual way (see Fig. 26), provided the natural period of vibration of the moving element is long in comparison with the frequency of the impressed voltage. This follows from the fact that the *average* torque, which determines the permanent deflection of the moving element, is equal to the *average* value of the product vi of the instantaneous voltage and current.

Since ordinary alternating currents pass through a complete cycle in a small fraction of a second, usually less than $\frac{1}{25}$ second, it is the average power, and not the instantaneous power, that one is usually interested in. Hence it has become the practice, in speaking of alternating current power, to omit the word average, and to say simply the power input, or power output, leaving the word average to be understood. This practice will be followed throughout the rest of this book, and whenever the expression "power input," or "power output" is used, it is to be understood, unless specifically stated otherwise, that the *average* power input or output is meant.

Problem 9.—Referring to the problems at the end of the preceding article, what is the average power input to the circuit described (*a*) in Problem 5? (*b*) In Problem 6? (*c*) In Problem 7. (*d*) in Problem 8?

Answer.—(*a*) 14.58 kilowatts. (*b*) 500 watts. (*c*) Zero. (*d*) Zero.

Problem 10.—(*a*) Plot to scale the instantaneous power supplied to an alternating-current motor when a sine-wave voltage of 220 volts (r.m.s. value) is impressed on its terminals, producing in it a current of 100 amperes (r.m.s. value) lagging the voltage by 60 degrees. (*b*) What is the power input to this motor?

Answer.—(*b*) 11 kilowatts.

168. Power Factor.—When the current and voltage are in phase ($\theta = 0$), the power input or output is equal simply to the product of the r.m.s. value of the voltage by the r.m.s. value of the current. Since an alternating-current voltmeter

reads the r.m.s. value of the voltage, and an alternating-current ammeter reads the r.m.s. value of the current, the product of the voltmeter and ammeter readings then gives the power input or output, just as in the case of a direct-current circuit.

However, *when there is a difference in phase between the current and voltage, as is usually the case,* **the product of the voltmeter and ammeter readings does NOT give the power.** When there is a phase difference, the power is always less than the product of the voltmeter and ammeter readings.

In general, the factor by which the product of the r.m.s. value of the terminal voltage of a circuit, and the r.m.s. value of the current in this circuit, must be multiplied, in order to give the true average power input or output of this circuit, is called the "power factor" of this circuit. That is, calling V the r.m.s. value of the terminal voltage in volts, I the r.m.s. value of the current in amperes, and P the average power input (or output) in watts, then the power factor is

$$\text{Power factor} = \frac{P}{VI} \tag{16}$$

The power factor of a circuit is always less than unity. Instead of expressing the power factor as a fraction, however, it is common practice to express it as a percentage, viz.,

$$\text{Per cent. power factor} = 100\,\frac{P}{VI} \tag{16a}$$

Comparing equation (16) with equation (15), it is evident that when the voltage and current are both sine waves, the power factor is equal to the cosine of the angle by which they differ in phase. This phase angle is therefore frequently called the "power-factor angle" of the circuit.

The simplest way of determining the power factor of a circuit is to measure simultaneously the r.m.s. value V of the impressed voltage, by means of an alternating-current voltmeter, the r.m.s. value I of the current, by means of an alternating-current ammeter, and the average power input (or output) P, by means of a wattmeter. The power factor is then calculated from equation (16*a*).

When the current and voltage in a circuit are in quadrature, *i.e.*, when they differ in phase by 90 degrees, and both are sine waves, the average power input is $VI \cos 90 = 0$. The

power factor of such a circuit is therefore zero. A voltmeter connected across the terminals of such a circuit will indicate the voltage, say 100 volts, and an ammeter connected in series with it will indicate the current, say 5 amperes, but a wattmeter connected to such a circuit will read zero watts.

A condenser whose leakage conductance is zero has a zero power factor, since the current taken by it leads the impressed voltage by 90 degrees (see Problem 8, Article 166). Similarly, a coil of high self-inductance and of low resistance has a power factor very nearly equal to zero, since the current through it lags the impressed voltage by nearly 90 degrees (see Problem 7, Article 166). A non-inductive resistance, on the other hand, has *practically* unity power factor, since the current through it is in phase with the impressed voltage (see Problem 6, Article 166).

Problem 11.—Referring to the problems at the end of Article 166, what is the power factor of the circuit described (*a*) in Problem 5? (*b*) In Problem (6)? (*c*) In Problem (7)? (*d*) In Problem (8)? (*e*) What is the power factor of the motor described in Problem 10, Article 167?

Answer.—(*a*) 0.972, or 97.2 per cent. (*b*) Unity or 100 per cent. (*c*) Zero. (*d*) Zero. (*e*) 50 per cent.

169. Apparent Power.—The product of the r.m.s value of the voltage impressed across the terminals of a circuit, by the r.m.s. value of the current in this circuit, is called the "apparent power input," or the "volt-amperes" input. Similarly, the volt-amperes output of a circuit which gives out electric energy is equal to the product of the r.m.s. value of its terminal voltage and the r.m.s. value of the current in it. Apparent power is also expressed in kilovolt-amperes, abbreviated "kv.-a.," or "*K.V.A.*" The kilovolt-amperes of a circuit is equal to the volt-amperes divided by 1000.

The heating of an alternating-current generator or transformer is due primarily to (1) the heating effect of the current, equal to rI^2, where I is the r.m.s. value of the current and r is the resistance of its path, and (2) to the heating effect due to the eddy currents and hysteresis in the magnetic circuit. The eddy-current and hysteresis losses depend upon the magnetic flux in the magnetic circuit, and this flux in turn is proportional to the electromotive force which is developed in the machine, which is approximately equal to its terminal voltage V. Hence the

total amount of heat developed in such an alternating-current machine depends upon the r.m.s. values of the current and voltage, and not upon the average power output or input. For this reason, alternating-current generators and transformers are usually rated in kilovolt-amperes, and not in kilowatts.

Problem 12.—Referring to the problems at the end of Article 166, what is the volt-ampere input to the circuit described (*a*) in Problem 5? (*b*) In Problem 6? (*c*) In Problem (7)? (*d*) In Problem (8)? (*e*) What is the kilovolt-ampere input to the motor described in Problem 10, Article 167.

Answer.—(*a*) 15,000 volt-amperes. (*b*) 500 volt-amperes. (*c*) 471 volt-amperes. (*d*) 754 volt-amperes. (*e*) 22 kilovolt-amperes.

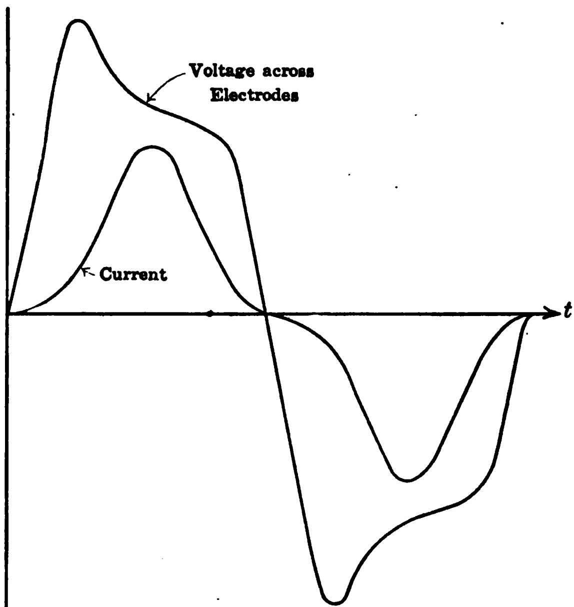


FIG. 130.—Current and voltage in A. C. arc.

170. Equivalent Phase Difference.—When the voltage and current in a circuit are not sine waves, the power input is usually less than the product of the r.m.s. values of the voltage and current, even though the latter pass through their zero values at the same time (see Article 211). That is, the power factor of a circuit to non-sinusoidal voltages and currents is usually less than unity.

For example, when a sine-wave voltage is impressed across the terminals of an arc lamp, the voltage drop across the arc from

one electrode to the other, and current through the arc, have wave shapes as shown in Fig. 130. The current passes through its zero value at the same instant that the voltage is zero. However, the power input to the arc is not equal to the product of the r.m.s. values of the current and voltage, but is only about 80 per cent. of this product.

In general, irrespective of the wave shapes of the voltage and current, the angle whose cosine is equal to the power factor (expressed as a fraction) is called the "equivalent" phase difference between the voltage and current. That is, calling V the r.m.s. value of the voltage, I the r.m.s. value of the current, and P the average power, the equivalent phase difference is

$$\theta = \cos^{-1}\left(\frac{P}{VI}\right) \tag{17}$$

By using this equivalent phase difference as the value of the angle θ, the various relations which hold strictly only for sine-wave voltages and currents may also be applied to non-sinusoidal voltages and currents, and, unless their wave forms are badly distorted, will, in most cases, lead to results sufficiently accurate for practical purposes. This method of analysis is equivalent to considering the actual voltage and current as equivalent to a sine-wave voltage and current of the same r.m.s. values respectively at their actual r.m.s. values, and differing in phase by the equivalent phase angle defined by equation (17). However, there are numerous cases when a more accurate analysis is required (see Chapter XVIII).

171. Addition of Alternating Quantities.—In practice, an alternating quantity, such as an alternating current or electromotive force, is always specified numerically by its r.m.s. value. That is, when one speaks of an alternating current of 10 amperes, what is invariably meant is an alternating current whose r.m.s. value is 10 amperes. Similarly, by an alternating electromotive force of 50 volts is meant an alternating electromotive force whose r.m.s. value is 50 volts.

Also, when the capital letters I, E, and V are used in any formula, the numerical values to be assigned to these letters, unless specifically stated otherwise, are the r.m.s. values of the quantities to which they refer, *i.e.*, to what the ammeters and voltmeters in the circuits indicate. This practice often confuses the beginner, for his natural tendency is to apply to the quanti-

ties represented by these symbols the same laws as hold for the quantities represented by these same symbols in direct-current circuits. This tendency must be carefully avoided, for, except in certain special cases, **the laws for direct currents do not apply to the r.m.s. values of alternating-current quantities.**

For example, the beginner is inclined to think that when an alternating-current generator supplies energy to two different circuits connected to its terminals (see Fig. 131), the r.m.s. value of the current I_s supplied by the generator is equal to the sum of the r.m.s. values of the currents I_1 and I_2 supplied to the two circuits. This, however, is never true, except in the special case when the currents taken by the two leads are

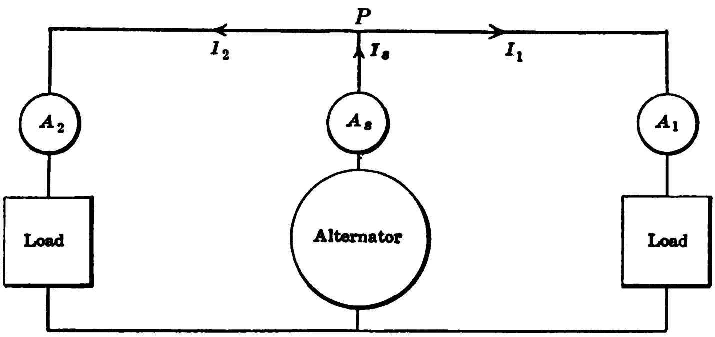


FIG. 131.—Alternator supplying two loads.

in phase. When these currents are not in phase (and they usually are not) the r.m.s. value of the generator current I_s is always less than the sum of the r.m.s. values of the load currents. That is, an ammeter A_s connected in one of the leads from the generator will read less than the sum of the readings of the ammeters A_1 and A_2 connected in series with the respective loads.

This may be readily seen by a simple graphical analysis. Assume the currents taken by the loads to be sine-wave currents, and let the current I_1 taken by load No. 1 lag the current I_2 taken by load No. 2 by the angle $\theta = \theta_1 - \theta_2$, as shown in Fig. 132. The arrows on the diagram in Fig. 131 indicate the direction which is chosen as the positive "sense" of these currents. These currents are, of course, actually in one direction during one-half of each cycle, and in the opposite direction during the other half of the cycle. The arrow on any particular conductor means simply that at any instant at which the cur-

rent in this conductor is actually in the direction of the arrow, its instantaneous value is to be considered as positive, and at any instant at which it is in the opposite direction, its instantaneous value is to be considered as negative. This convention is a general one, and is followed throughout this book.

With this understanding, the instantaneous values of the currents I_1 and I_2 supplied to the two loads may be represented by the curves in Fig. 132 whose maximum values are I_{1m} and

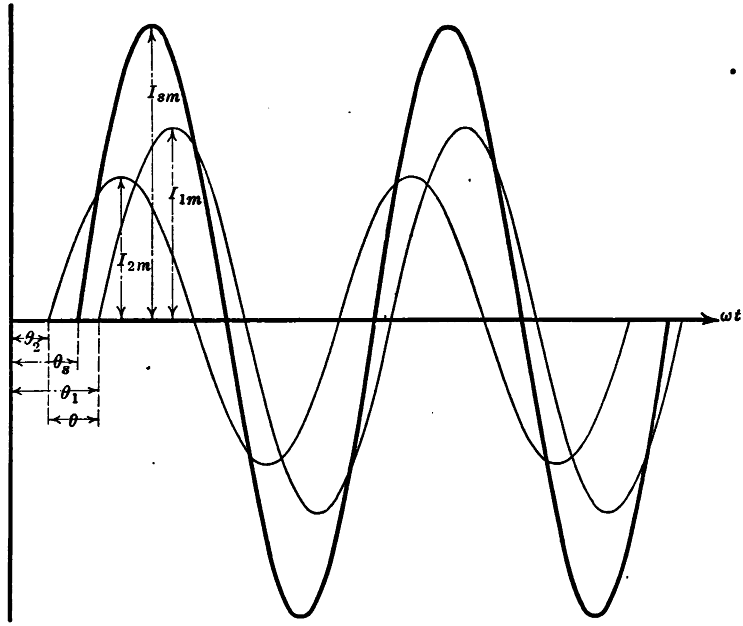


FIG. 132.—Resultant of two sine-wave currents.

I_{2m} respectively. Since *at each instant* the total current which enters the junction P in Fig. 131 must be equal to the current which leaves this junction (neglecting any electric charge on the surface of the conductors at this junction), the value, *at any particular instant,* of the generator current must be equal to the algebraic sum of the ordinates, corresponding to this instant, of the curves which represent I_1 and I_2. Hence the instantaneous values of the generator current are given by the ordinates of the heavy curve in Fig. 132, which curve is constructed by adding algebraically the ordinates of the other two curves.

The heavy curve which represents the instantaneous values

of the generator current is also a sine curve, as shown in the figure (a rigorous proof is given below). It may also be seen by inspection that the maximum value I_{sm} of this curve is less than the sum of the maximum values I_{1m} and I_{2m} of the other two curves.

Hence, since the r.m.s. value of each of these sine curves is equal to its maximum value divided by $\sqrt{2}$, it follows that the r.m.s. value of the generator current must be less than the sum of the r.m.s. values of the two load currents.

In general, as will be shown in the next section, whenever an alternating current divides into two currents, or whenever two alternating currents combine to produce a single current, the r.m.s. value of the single, or resultant, current is always less than the sum of the r.m.s. value of the two component currents, except when the two component currents are in phase. Similarly, the r.m.s. value of the resultant of two alternating electromotive forces is always less than the sum of the r.m.s. values of these two component electromotive forces, unless these two component electromotive forces are in phase. Moreover, as illustrated in Fig. 132, the resultant of two sine-wave quantities which differ in phase is, in general, in phase with neither of these quantities.

Problem 13.—A 25-cycle, single-phase generator (*i.e.*, an alternator whose armature winding forms a single circuit, with two slip rings) supplies a current of 100 amperes to a single-phase induction motor, and a current of 50 amperes to a single-phase synchronous motor. The power factor of the induction motor is 70 per cent., and the current taken by it lags the pertage drop through it. The power factor of the synchronous motor is 5 vol cent., and the current leads the voltage drop through it. The resistance and inductance of the leads between the motors and the generator may be neglected.

(*a*) What is the difference in phase between the currents supplied to the two motors? (*b*) Plot these two currents to scale, taking 1 inch along the horizontal axis equal to 50 degrees, and 1 inch along the vertical axis equal to 50 amperes. (*c*) Plot the instantaneous values of the generator current. (*d*) What is the r.m.s. value of the generator current? (*e*) What is the difference in phase between the generator current and the generator terminal voltage? (*f*) What is the power factor of the generator, *i.e.*, of the total load supplied by the generator? (*g*) If this terminal voltage is 2200 volts, what is the power input to each motor? (*h*) What is the power output of the generator? Compare with the total power input to the motors. (*i*) What is the kv.-a. input to each motor? (*j*) What is the kv.-a. output of the generator? Compare with the total kv.-a. input to the motors.

Answer.—(*a*) 132.7 degrees. (*d*) 75.8 amperes. (*e*) Generator current lags the terminal voltage by 17.5 degrees. (*f*) 95.9 per cent. (*g*) 154 kilowatts to the induction motor and 5.5 kilowatts to the synchronous motor. (*h*) 159.5 kilowatts, equal to the sum of the power inputs to the two motors. (*i*) 220 kv.-a. to the induction motor and 110 kv.-a. to the synchronous motor. (*j*) 167 kv.-a., which is 49.4 per cent. *less than* the total kv.-a., input to the two motors.

172. Vector Representation of Sine-wave Quantities.—The determination of the resultant of two or more alternating quantities by plotting their instantaneous values is a tedious process. When the given quantities are sine-wave quantities, both the r.m.s. value and the phase angle of their resultant

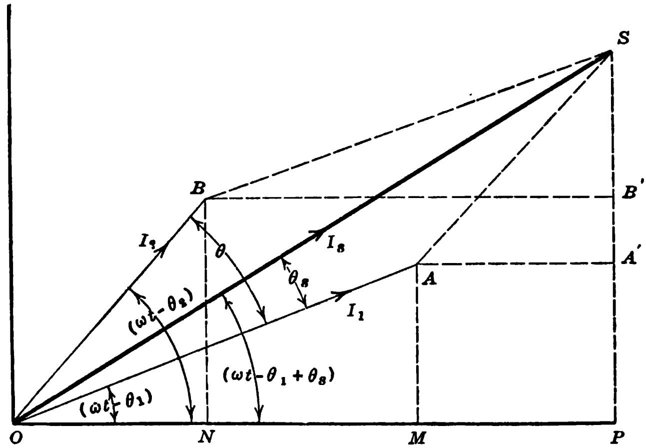


FIG. 133.—Vector representation of two sine-wave currents.

may be determined by a much simpler and shorter method. The principle involved in this method is that any sine-wave quantity may be represented by a vector whose length is equal to the r.m.s. value of this quantity, and which makes with a given reference line an angle equal to the phase angle of this quantity with respect to any arbitrarily chosen instant of time.

For example, the instantaneous values of two currents, whose r.m.s. values are I_1 and I_2, and whose phase angles referred to any given instant of time are θ_1 and θ_2 (see Fig. 132), are given respectively by the equations

$$i_1 = \sqrt{2}\, I_1 \sin (\omega t - \theta_1)$$
$$i_2 = \sqrt{2}\, I_2 \sin (\omega t - \theta_2)$$

where $\omega = 2\pi f$, and f is the frequency.

In the polar diagram in Fig. 133, the line $\overline{OA}$ is laid off equal

in length to the r.m.s. value I_1 and at an angle $(\omega t - \theta_1)$ with the horizontal axis OX. The vertical distance from A to this horizontal axis, namely, the distance $\overline{AM}$, is then

$$\overline{AM} = I_1 \sin (\omega t - \theta_1)$$

and therefore

$$i_1 = \sqrt{2}\,\overline{AM}$$

Similarly, the line $\overline{OB}$ is made equal to the r.m.s. value I_2 of the second current, and the angle between this line and the horizontal axis OX is made equal to $(\omega t - \theta_2)$. The length of the line $\overline{BN}$ is then

$$\overline{BN} = I_2 \sin (\omega t - \theta_2)$$

and therefore

$$i_2 = \sqrt{2}\,\overline{BN}$$

As time increases, the angle ωt increases proportionally, and therefore the lines $\overline{OA}$ and $\overline{OB}$ rotate at a constant speed of ω radians per second, *i.e.*, make one complete revolution per cycle. In this diagram, as in all other vector diagrams in this book, the positive direction of rotation is taken as the **counter-clockwise direction.** This is the convention adopted by the International Electro-technical Commission.

As the two vectors $\overline{OA}$ and $\overline{OB}$ rotate, the lengths of the lines $\overline{AM}$ and $\overline{BN}$ change, being at each instant equal respectively to the values of the two currents at this instant divided by $\sqrt{2}$. The value of the sum of the two currents i_1 and i_2 at any instant, which may be represented by the symbol i, is therefore

$$i = i_1 + i_2 = \sqrt{2}\,(\overline{AM} + \overline{BN}) \qquad (18)$$

But $(\overline{AM} + \overline{BN})$ is equal at each instant to the vertical line $\overline{SP}$, where $\overline{OS}$ is the diagonal of the parallelogram $OASB$. This parallelogram remains fixed in shape as the two lines $\overline{OA}$ and $\overline{OB}$ rotate, for these lines, being equal in length to the r.m.s. values of the two currents, are of *fixed length,* and the angle between them, namely, the angle

$$\theta = \omega t - \theta_2 - (\omega t - \theta_1) = \theta_1 - \theta_2$$

is likewise *constant,* since this angle is equal to the difference in phase between the two currrents.

Hence the line $\overline{OS}$ remains fixed in length, rotates with a constant velocity ω, and makes with the line $\overline{OA}$ a constant angle a_s. The line $\overline{OS}$, therefore, makes with the reference line OX θn angle

$$\omega t - \theta_1 + \theta_s$$

Consequently, the length of the line $\overline{SP}$ at any instant is

$$\overline{SP} = \overline{OS} \sin (\omega t - \theta_1 + \theta_s)$$

But $\overline{SP} = \overline{SB'} + \overline{B'P} = \overline{AM} + \overline{BN}$. Whence, from equation (18), the instantaneous value of the resultant current i is

$$i = \sqrt{2}\, \overline{OS} \sin (\omega t - \theta_1 + \theta_s) \tag{18a}$$

Note that $\overline{OS}$ is numerically equal to the *vector sum* of $\overline{OA}$ and $\overline{OB}$ (see Article 2). Hence

(1) The resultant of two sine-wave quantities is likewise a sine-wave quantity,

(2) The r.m.s. value I_s of the resultant of two sine-wave quantities is the sum of two vectors whose lengths are respectively the r.m.s. values I_1 and I_2 of the two given quantities, and which make with each other an angle equal to the difference in phase θ between these two quantities.

Comparing equation (18*a*) with the expression

$$i_1 = \sqrt{2}\; I_1 \sin (\omega t - \theta_1)$$

it is evident that the resultant of the two given sine-wave quantities *leads* this particular component by the angle θ_s, which is a constant angle, since $\overline{OS}$ is fixed with respect to $\overline{OA}$, although both rotate with the angular velocity ω.

Consequently, as far as the r.m.s. value and phase angle of the resultant of two sine-wave quantities is concerned, the vectors which represent these two quantities may be considered as *stationary* vectors. The vectors which represent the component and resultant quantities need be considered as rotating only when one wishes to consider the *instantaneous* values of these quantities. In all vector diagrams used hereafter the vectors will be considered as stationary, unless specifically stated otherwise. Moreover, in any given problem the vector representing a *given* current or voltage will always be taken as the reference line for the diagram (*e.g.*, the line $\overline{AT}$ in Fig. 6, Article 2).

Note that the relations just stated apply to any kind of sine-wave quantities, whether currents, voltages, magnetic fluxes, or any other quantity. Hence, by applying the laws of vector addition and subtraction (see Article 2, which should be carefully re-read at this point), any problem involving only the r.m.s. values and phase differences of sine-wave quantities may always

be solved without considering the instantaneous values of these quantities.

Problem 14.—(*a*) What is the r.m.s. value of the voltage required to establish a 60-cycle current of 10 amperes in a coil whose resistance is 2 ohms and whose self-inductance is 0.1 henry? (Note that such a coil is equivalent of a non-inductive resistance of 2 ohms in series with a coil having zero resistance and an inductance of 0.1 henry; see Article 126.) (*b*) Draw to scale a vector diagram showing the voltage drop in the coil due to its resistance, and the voltage drop due to its inductance, and the resultant voltage drop. (*c*) Does the voltage drop through the coil lead or lag the current, and by what angle? (*d*) What is the power factor of this coil for a 60-cycle current? (*e*) What would be the value of the voltage required to

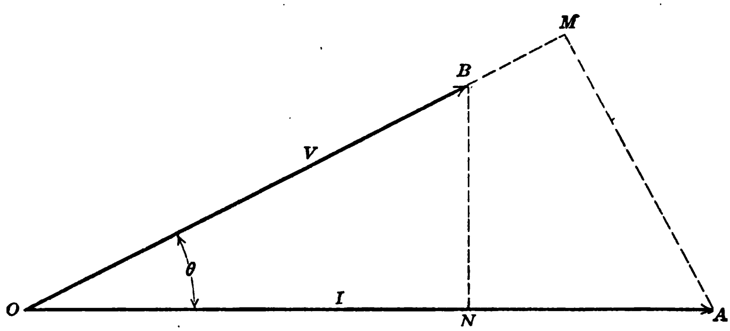


FIG. 134.—Active and reactive components.

establish a 25-cycle current of 10 amperes in this coil? (*f*) What would be the value of the voltage required to establish a direct current of 10 amperes in the coil?

Answer.—(*a*) 377 volts. (*c*) The voltage drop leads the current by an angle of 87.0 degrees. (*d*) 5.3 per cent. (*e*) 158.5 volts. (*f*) 20 volts.

173. Active and Reactive Components.—Consider any circuit in which a sine-wave impressed voltage produces a sine-wave current. Let V be the r.m.s. value of this voltage, let I be the r.m.s. value of the current, and let θ be the angle by which the current lags the voltage. The vector diagram of this voltage and current is then taken as shown in Fig. 134. The vector $\overline{OA}$ is equal in length to the current I, and the vector $\overline{OB}$ equal in length to the voltage V.

From the diagram it is evident that the component of the current vector $\overline{OA}$ in the direction of the voltage vector $\overline{OB}$ is

$$\overline{OM} = \overline{OA}\cos\theta = I\cos\theta \qquad (19)$$

Similarly, the component of the voltage vector $\overline{OB}$ in the direction of the current vector $\overline{OA}$ is

$$\overline{ON} = \overline{OB}\cos\theta = V\cos\theta \qquad (19a)$$

As shown in Article 167, the power input to such a circuit is

$$P = V(I\cos\theta) = I(V\cos\theta)$$

Hence the power input is equal to the voltage impressed on the circuit multiplied by the *component* of the current *in phase* with this voltage. Similarly, the power input to a cirucit is also equal to the current through the circuit multiplied by the *component* of the impressed voltage in phase with this current.

From these relations, the component of the current in a circuit in phase with the voltage impressed on it is sometimes called the "power component" of the current. Similarly the component of the voltage in phase with the current is called the "power component" of the voltage. The other component of the current, viz., the component

$$\overline{AM} = I\sin\theta \qquad (20)$$

which is at right angles to the voltage, is sometimes called the "wattless component" of the current, and the other component of the voltage, viz., the component

$$\overline{BH} = V\sin\theta \qquad (20a)$$

is sometimes called the "wattless component" of the voltage.

These names "power component" and "wattless component" are, however, misleading, in that both components represent power. The difference between the two is that the power corresponding to the in-phase component has an *average* value equal to the average power input to the circuit, whereas the power corresponding to the quadrature component has an *average* value equal to zero. The power corresponding to the in-phase component is equal to the average rate at which the electric energy transferred to the circuit is converted into heat or mechanical work. The energy corresponding to the quadrature component is the energy of the magnetic and electrostatic fields which surround the conductors. Since the energy of these fields is the same at the end of every cycle, its algebraic mean rate of transfer to or from the circuit, *i.e.*, its average power, is zero.

More significant names for the two components of the current and voltage are therefore the "active component" and "reactive component." These are the terms recommended by the American Institute of Electrical Engineers. The *active* component of the current in a circuit is that component of this current which is *in phase* with the terminal voltage of the circuit, and the *reactive* component is that component which is in quadrature with the terminal voltage; similarly for the two components of the terminal voltage with respect to the current.

Problem 15.—A single-phase induction motor connected to 220 volt mains takes a current of 50 amperes when it develops 10 horsepower. The efficiency of the motor is 90 per cent. (The current taken by an induction motor always lags the impressed voltage.)

(*a*) What is the power input to the motor? (*b*) What is the kv.-a. input? (*c*) What is the difference in phase between the voltage and current? (*d*) Draw the vector diagram. (*e*) What is the active component of the current? (*f*) What is the reactive component of the current? (*g*) What is the active component of the voltage? (*h*) What is the reactive component of the voltage?

Answer.—(*a*) 8.29 kw. (*b*) 11.0 kv.-a. (*c*) 41.1 degrees. (*e*) 37 7 amperes. (*f*) 32.9 amperes. (*g*) 165.8 volts. (*h*) 144.6 volts.

XV

IMPEDANCE AND ADMITTANCE, REACTANCE AND SUSCEPTANCE, EFFECTIVE RESISTANCE AND EFFECTIVE CONDUCTANCE

174. Introduction.—The distribution of alternating currents in a network depends not only upon the resistance of each part of the network, but also upon the inductance and electrostatic capacity of each part of the network. The calculation of the alternating currents in the various branches of a network is therefore somewhat more complicated than similar calculations for direct currents. However, such calculations may be readily affected in terms of certain "circuit constants" which will now be defined, and whose values in terms of resistance, inductance and electrostatic capacity will be given.

In the calculation of series circuits, the circuit constants usually employed are the impedance, reactance, and effective resistance of each branch. In the calculation of parallel circuits, the circuit constants usually employed are the admittance, susceptance and effective conductance of each branch.

These circuit constants take into account only the energy which is dissipated as heat and that which surges back and forth from the electric circuit to the magnetic and electric fields within and surrounding the circuit. When there is a transformation of electric energy into mechanical work, or conversely, or when there is a transfer of electric energy from one circuit to another by mutual induction, such transformation or transfer of energy is most conveniently expressed in terms of the electromotive forces which produce or which oppose it.

175. Reactance and Impedance of a Coil to a Sine-wave Current.—As shown in Article 126, the voltage drop through a coil of constant resistance r and constant self-inductance L (see Fig. 135), when the only source of electromotive force in this coil is the varying magnetic field due to the current i in it, is

$$v = ri + L\frac{di}{dt} \tag{1}$$

where i is the value of the current at the particular instant under consideration. The voltage drop v given by this expression is through the coil in the direction of the current i.

In the special case when the current is a sine-wave current of frequency f and r.m.s. value I, its value at each instant may be written

$$i = \sqrt{2}\, I_e \sin (2\pi ft - \theta_i) \qquad (2)$$

where θ_i is the phase angle of this current referred to any arbitrarily chosen instant of time.

The voltage drop through the coil, in the direction of the current, *due solely to its resistance*, is then

$$ri = \sqrt{2}\, (rI) \sin (2\pi ft - \theta_i)$$

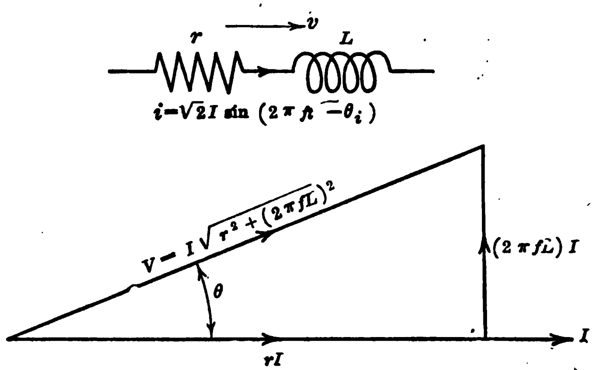


FIG. 135.—Vector diagram for a coil.

Comparing this expression with equation (2), it is seen that the r.m.s. value of this resistance drop is equal to the product of the resistance r by the r.m.s. value I of the current, and is *in phase* with the current.

The voltage drop through the coil, *due solely to its inductance*, is

$$L\frac{di}{dt} = \sqrt{2}\, (2\pi fLI) \cos (2\pi ft - \theta_i)$$

$$= \sqrt{2}\, (2\pi fLI) \sin \left(2\pi ft - \theta_i + \frac{\pi}{2}\right)$$

Comparing this expression with equation (2), it is seen that the r.m.s. value of this "inductive" drop is equal to the product

of the quantity $(2\pi fL)$ by the r.m.s. value I of the current, and *leads the current by* $\frac{\pi}{2}$ *radians, or* 90 *degrees.*

The total voltage drop through the coil is then equal to the vector sum of two vectors of lengths rI and $(2\pi fLI)$, and which differ in phase by 90 degrees, as shown in Fig. 135. The r.m.s. value of the resultant voltage drop through the coil is therefore

$$V = I\sqrt{r^2 + (2\pi fL)^2} \tag{3}$$

and this voltage drop (in the direction of the current in the coil) leads the current by the angle

$$\theta = \tan^{-1}\frac{2\pi fL}{r} \tag{4}$$

The quantity $(2\pi fL)$, namely, the self-inductance of the coil multiplied by 2π times the frequency of the current established through it, is called the "reactance" of the coil. Reactance is usually represented by the smybol x. Hence the reactance of a coil of constant self-inductance L, to a sine-wave current whose frequency is f cycles per second, is

$$x = 2\pi fL \tag{5}$$

The product of the reactance x by the r.m.s. value of the current I in the coil, viz., the product xI, is then equal to the r.m.s. value of the drop of voltage through coil due to its inductance. This component of the total voltage drop through the coil is therefore commonly referred to as the "reactance drop" in the coil. The other component of the total voltage drop, viz., the component rI, is called the "resistance drop." Note particularly that the resistance drop is in phase with the current, whereas *the reactance drop leads the current by* 90 *degrees.*

The student should note carefully that, although the self-inductance of a coil may be strictly a constant, its *reactance, and therefore the reactance drop, varies directly as the frequency of the current.* In fact, a coil which has practically negligible reactance to low-frequency currents (25 or 60 cycles per second) will have a relatively enormous value to the high-frequency currents (100,000 cycles per second or more) used in wireless telegraphy. Also note that, except as the intensity of the current may affect the reluctance of the magnetic circuit of the coil, the reactance of a coil is independent of the r.m.s. value of the current, just as its resistance is independent of the current strength (provided the temperature remains constant).

In terms of the reactance x, the r.m.s. value of the total voltage drop, and the angle θ by which the voltage drop leads the current, may be written

$$V = I\sqrt{r^2 + x^2} \tag{6}$$

and

$$\theta = \tan^{-1}\frac{x}{r} \tag{7}$$

The quantity $\sqrt{r^2 + x^2}$ is called the "impedance" of the coil. Impedance is usually represented by the symbol z. Hence the impedance of a coil of resistance r and reactance x is

$$z = \sqrt{r^2 + x^2} \tag{8}$$

Note from equation (6) that the impedance of a coil is also equal to the ratio of the r.m.s. value of the voltage across its terminals to the r.m.s. value of the current through it. The impedance of a coil to a current of a given frequency is therefore readily determined experimentally by impressing across its terminals a sine-wave voltage of the given frequency, and noting the reading V of a voltmeter connected across its terminals and the reading I of an ammeter in series with it. The impedance is then

$$z = \frac{V}{I} \tag{8a}$$

Both reactance and impedance are of the same "dimensions" as resistance, viz., potential difference divided by current. The practical unit of reactance and impedance is therefore the ohm.

From Fig. 135 it is evident that the resistance drop rI is the active component of the impressed voltage V. Hence the average power input to the coil is

$$P = (rI) \times I = rI^2 \tag{9}$$

This power input is identical with the power input corresponding to an equal current I in a non-inductive resistance r. The energy corresponding to this power all goes into heat. Note that this relation is identical with Joule's Law for a direct current (see Article 35). In fact, irrespective of the wave form of an alternating current, the average rate at which heat is developed by it in a conductor of resistance r is always rI^2, where I is the r.m.s. value of the current.

Due to the eddy currents produced in a conductor by the varying flux which accompanies an alternating current, the dis-

tribution of an alternating current in a conductor is in general different from the distribution of a direct current in this conductor. On this account, the resistance of a conductor to an alternating current is in general greater than its resistance to a direct current. However, equation (9) may always be used as a basis for the experimental determination of the alternating-current resistance of a coil, for the average power input to the coil may always be measured by means of a wattmeter. That is, calling P the wattmeter reading, then the resistance of the coil to the given alternating current is

$$r = \frac{P}{I^2} \tag{9a}$$

(NOTE: This expression for the alternating-current resistance of a coil is applicable only when the total energy input to the coil appears as heat energy (see Article 178)).

The difference between the alternating-current resistance, or "effective" resistance, of a conductor and its direct-current, or "ohmic" resistance, is usually inappreciable for ordinary sizes of wire (No. 0000 A. W. G. and smaller), provided the frequency is less than 60 cycles per second, and there is no iron in the magnetic field produced by the current. Under these conditions the alternating-current resistance of a conductor, whether straight or wound into a coil, may be taken equal to its ohmic resistance.

When both the impedance and resistance of a coil to a sine-wave current of a given frequency are known, its reactance at this frequency may be calculated from the formula

$$x = \sqrt{z^2 - r^2} \tag{10}$$

The self-inductance of the coil may then be found by dividing this reactance by 2π times the frequency f.

Problem 1.—A coil has a constant resistance of 5 ohms and a constant self-inductance of 0.02 henry.

(*a*) What is the reactance of this coil to a 60-cycle current? (*b*) To a 25-cycle current? (*c*) What is its impedance to a 60-cycle current? (*d*) To a 25-cycle current? (*e*) What is the power factor of the coil to a 60-cycle current? (*f*) To a 25-cycle current. (*g*) An alternating current whose r.m.s. value is 3 amperes is established through the coil. What is the average rate at which heat is dissipated in it? (*h*) Does this average rate of dissipation of heat depend on the wave form or frequency, assuming the resistance to be constant?

Answer.—(*a*) 7.54 ohms. (*b*) 3.14 ohms. (*c*) 9.04 ohms. (*d*) 5.90 ohms. (*e*) 55.3 per cent. (*f*) 84.7 per cent. (*g*) 45 watts. (*h*) No.

Problem 2.—A certain two-wire transmission line is 50 miles long. The wires are No. 000 A.W.G. stranded copper, and are spaced 4 feet between centers.

(*a*) What is the total resistance of this line? (*b*) What is the total self-inductance of the loop formed by the two wires? (*c*) What is the total reactance of the line to a 60-cycle sine-wave current? (*d*) What is the total impedance of the line to this current. (*e*) What will be the total voltage drop in both wires of the line when the current is 75 amperes? (*f*) What will be the difference in phase between this voltage drop and the current?

Answer.—(*a*) 33.9 ohms. (*b*) 0.184 henry. (*c*) 69.3 ohms. (*d*) 77.1 ohms. (*e*) 5780 volts. (*f*) The voltage drop leads the current by 64 degrees.

Problem 3.—A certain air-core solenoid is connected to a source of 60-cycle, sine-wave voltage, and a voltmeter, ammeter, and wattmeter are connected to read the voltage impressed on the coil, the current taken by it, and the power absorbed by it. The readings of the three instruments are respectively 105 volts, 7.9 amperes, and 320 watts.

(*a*) Draw a diagram showing the connections of the instruments. (*b*) What is the impedance of the coil? (*c*) Its resistance? (*d*) Its reactance? (*e*) Its self-inductance? (*f*) Draw a vector diagram showing the phase relation of the current and voltage.

Answer.—(*b*) 13.3 ohms. (*c*) 5.12 ohms. (*d*) 12.3 ohms. (*e*) 0.0326 henry.

176. Susceptance and Admittance of a Condenser to a Sine-wave Voltage.—As shown in Article 156, the total current which flows into the A plate of a condenser of constant capacity C and constant leakage conductance g (see Fig. 136), when a drop of potential v is established through it from its A plate to its B plate, is

$$i = gv + C\frac{dv}{dt} \tag{11}$$

The component of the current represented by the term gv is the leakage current of the condenser, and the component represented by the term $C\frac{dv}{dt}$ is the charging current.

In the special case when the voltage across the condenser is a sine-wave voltage of frequency f and r.m.s. value V, its value at any instant may be written

$$v = \sqrt{2}\ V \sin (2\pi ft - \theta_v) \tag{12}$$

where θ_v is the phase angle of this voltage referred to any arbitrarily chosen instant of time.

The leakage current through the condenser, in the direction of the voltage drop, is then

$$gv = \sqrt{2}\ (gV)\ \sin\ (2\pi ft - \theta_v)$$

Comparing this expression with equation (12), it is evident that the r.m.s. value of the leakage current is equal to the product of the leakage conductance g by the r.m.s. value V of the impressed voltage, and is *in phase* with this voltage.

The charging current of the condenser is

$$C\frac{dv}{dt} = \sqrt{2}\ (2\pi fCV)\ \cos\ (2\pi ft - \theta_v)$$

The r.m.s. value of the charging current is therefore equal to the

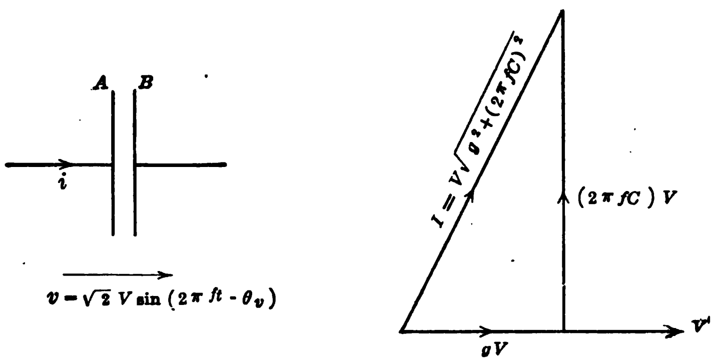


FIG. 136.—Vector diagram for a condenser.

product of the quantity $(2\pi fC)$ by the r.m.s. value of V of the impressed voltage, and *leads this voltage by* $\frac{\pi}{2}$ *radians or* 90 *degrees.*

The total current which flows into the condenser is then equal to the vector sum of two vectors of lengths gV and $(2\pi fCV)$, and which differ in phase by 90 degrees, as shown in Fig. 136. The r.m.s. value of this total current is therefore

$$I = V\ \sqrt{g^2 + (2\pi fC)^2} \qquad (13)$$

and this current leads the voltage drop through the condenser by the angle

$$\theta = \tan^{-1}\frac{2\pi fC}{g} \qquad (14)$$

The quantity $(2\pi fC)$, namely, the capacity of the condenser multiplied by 2π times the frequency of the impressed voltage, is called the "susceptance" of the condenser. Susceptance is usually represented by the symbol b. Hence the susceptance of a condenser of constant capacity C, to a sine-wave voltage whose frequency is f cycles per second, is[1]

$$b = 2\pi fC \tag{15}$$

The product of the susceptance b by the r.m.s. value V of the voltage impressed on the condenser, viz., the product bV, is then equal to the r.m.s. value of the charging current of the condenser, just as the product gV gives the r.m.s. value of the leakage current. Note particularly that the leakage current is in phase with the impressed voltage, whereas *the charging current leads the impressed voltage by 90 degrees.*

The student should note carefully that, although the capacity of a condenser may be strictly a constant, its *susceptance, and therefore its charging current, varies directly as the frequency of the impressed voltage.* In fact, a condenser whose charging current is practically inappreciable at low frequencies (25 or 60 cycles per second) will take a relatively very large charging current when the frequency is of the order of magnitude (100,000 cycles per second or more) of those used in wireless telegraphy.

In terms of the susceptance b, the r.m.s. value I of the total current taken by the condenser, and the angle θ by which this current leads the impressed voltage, may be written

$$I = V\sqrt{g^2 + b^2} \tag{16}$$

and

$$\theta = \tan^{-1}\frac{b}{g} \tag{17}$$

The quantity $\sqrt{g^2 + b^2}$ is called the "admittance" of the condenser. Admittance is usually represented by the symbol y. Hence the admittance of a condenser whose leakage conductance is g and whose capacity is C is

$$y = \sqrt{g^2 + b^2} \tag{18}$$

[1] This is the *capacity* susceptance of the condenser. The *effective* susceptance of a condenser is

$$b = -2\pi fC$$

(See Article 179.)

Note from equation (16) that the admittance of a condenser is also equal to the ratio of the r.m.s. value of the total current taken by it to the r.m.s. value of the voltage impressed across its terminals. The admittance of a condenser to a voltage of given frequency is therefore readily determined experimentally by measuring the r.m.s. value I of the current taken by the condenser when a voltage of this frequency and known r.m.s. value V is impressed across its terminals. The admittance is then given by the formula

$$y = \frac{I}{V} \tag{18a}$$

Both susceptance and admittance have the same "dimensions" as conductance, viz., current divided by potential difference. The practical unit of susceptance and admittance is therefore the mho.

From Fig. 136 it is evident that the leakage current gV is the active component of the total current I. Hence the average power input to the condenser is

$$P = V \times (gV) = gV^2 \tag{19}$$

This power all appears as heat, since the energy stored in the electric field of the condenser during one quarter of each cycle (while the voltage is increasing) is given back during the next quarter-cycle (while the field is decreasing).

As noted in Article 155 when an alternating voltage is impressed on a condenser whose dielectric is not perfectly homogeneous, the energy dissipated in the dielectric as heat is greater than that which would be dissipated in it due to the leakage current established through it by an equal constant voltage. This fact is most conveniently taken into account by considering the conductance of a condenser to an alternating voltage as greater than the conductance to a constant voltage, and to use the relation expressed by equation (19) as the basis for the definition of its "effective" conductance. That is, calling P the average power input to the condenser, as read by a wattmeter, when a sine-wave voltage of given frequency and given r.m.s. value V is impressed across its terminals, the effective conductance of the condenser is

$$g = \frac{P}{V^2} \tag{19a}$$

Compare with the corresponding expression, equation (9*a*), for the effective resistance of a coil.

When both the admittance and effective conductance of a condenser to a sine-wave voltage are known, its susceptance at this frequency may be calculated from the formula

$$b = \sqrt{y^2 - g^2} \tag{20}$$

The capacity of the condenser may then be found by dividing this susceptance by 2π times the frequency.

Problem 4.—(*a*) What is the capacity of the condenser formed by the two wires of the transmission line described in Problem 2 of the last article? (*b*) What is the susceptance of this condenser to a 60-cycle sine-wave voltage? (*c*) What is the r.m.s. value of the charging current taken by the line if the voltage between the wires has an average r.m.s. value of 45,000 volts?

Answer.—(*a*) 0.42 microfarads. (*b*) 0.000158 mho. (*c*) 7.1 amperes.

Problem 5.—A certain condenser is connected to a source of 60-cycle, sine-wave voltage, and a voltmeter, ammeter and wattmeter are connected to read the voltage impressed on it, the current taken by it, and the power absorbed by it. The readings of the three instruments are respectively 1200 volts, 1.7 amperes, and 78 watts.

(*a*) What is the admittance of the condenser? (*b*) Its effective conductance? (*c*) Its susceptance? (*d*) Its capacity? Draw a vector diagram showing the phase relation of the current and voltage.

Answer.—(*a*) 1416 micromhos. (*b*) 54.1 micromhos. (*c*) 1416 micromhos. (*d*) 3.76 microfarads.

177. Skin-effect.—As has been noted several times, the distribution of an alternating current in a conductor is in general different from that of a direct current. This phenomenon is due to the unequal electromotive forces set up in the different stream lines of the current by the varying magnetic flux *within the substance of the conductor* (see Article 118). In the particular case of a conductor in which the only appreciable magnetic field is that due to the current within this particular conductor, the effect of this internal varying flux is usually referred to as the "skin effect." This name arises from the fact that the magnetic flux tends to drive the current out toward the surface, or "skin," of the conductor.

Consider the particular case of a cylindrical conductor, *e.g.*, around wire. As shown in Article 98, the magnetic lines of force within such a conductor due to the current in it are circles concentric with its axis. The axis of the conductor is therefore

linked by all the internal flux. A stream line in the surface of the conductor is linked by none of this internal flux. Hence when the current in the conductor varies, the back electromotive force induced in the axis of the conductor is greater than the back electromotive force induced in the stream lines near its surface. Consequently, a given potential difference established between the ends of the conductor will cause less current to flow through unit area at the axis of the conductor than will flow through unit area near its surface.

The increase in the resistance of a cylindrical conductor due to this skin effect may be calculated, but the calculation is too complicated to be given here. See the article on *Skin Effect* in Pender's *Handbook for Electrical Engineers*. The effect increases both with the frequency and cross-section of the conductor, and is less for conductors of low conductivity than for those of high conductivity, *e.g.*, is less for aluminum than for copper. The skin effect in an ordinary stranded cable is practically the same as in a solid wire of the same size. Due to their high magnetic permeability, the effect in iron and steel conductors is much more pronounced than in non-magnetic conductors.

The magnitude of the skin effect, and its dependence upon the frequency and cross-section of the conductor, is illustrated by the following values for the ratio of the alternating current resistance to the direct resistance of a straight, round copper wire in which the *internal* magnetic field is due solely to the current in this particular wire. This condition is closely approximated in an overhead transmission line.

RATIO OF ALTERNATING-CURRENT TO DIRECT-CURRENT RESISTANCE; STRAIGHT, ROUND, COPPER WIRE

Size of Wire	Cycles per second					
	25	60	500	1000	100,000	1,000,000
No. 10 A. W. G.	1.00	1.00	1.00	1.00	3.02	10.0
No. 0 A. W. G.	1.00	1.00	1.07	1.25	10.0	31.3
No. 0000 A. W. G.	1.00	1.00	1.24	1.63	14.0	43.6
500,000 CM	1.00	1.02	1.76	2.38	21.3	67.0
1,000,000 CM	1.02	1.09	2.38	3.25	30.0	94.5

The skin effect also causes the self-inductance of a conductor to an alternating current to be less than its self-inductance to a

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direct current, but the variation in inductance is much less than the variation in resistance, and may usually be neglected.

178. Effective Resistance, Effective Reactance, and Impedance. —The definitions of reactance and impedance given in Article 175, as there stated, apply only to the special case of a coil. These terms are, however, used in a much more general sense, for they may be readily defined in such a manner as to be applicable to any portion of an electric circuit, whether a coil or a condenser, or to a combination of coils and condensers. These general definitions are applicable not only to actual sine-wave currents and voltages, but also to the sine-wave voltages and currents *equivalent to* non-sinusoidal currents and voltages (see Article 170).

In general, the varying magnetic flux which accompanies an alternating current in a given electric circuit not only tends to set up eddy currents in the conductors which form this circuit, but also in every other conductor in its vicinity. Both the unequal distribution of current produced thereby, and the actual eddy currents produced in the other conductors (such as an iron core) in the path of the flux, cause a dissipation of heat energy in excess of that which would be produced by an equal direct current.

Also, when there is iron or other ferromagnetic substances in the magnetic field established by the current, a certain amount of heat energy is dissipated in the iron, due to hysteresis. Again, a varying electric field in a dielectric is accompanied by a dissipation of heat energy due not only to the direct-current resistance of this dielectric, but also to dielectric hysteresis.

The total dissipation of heat energy due to all these causes may be conveniently taken into account by considering each portion of a circuit (*e.g.*, coil or condenser) as having an "effective" resistance of such a value, that the product of this resistance by the square of the current in the circuit is equal to the average rate at which heat energy is dissipated in this particular portion of the circuit and in the magnetic and electric fields within and surrounding it. This idea may be expressed mathematically as follows: Let I be the r.m.s. value of the total current in any given portion of a circuit, and let P_h the average rate at which heat energy is developed (1) in the conductors which form this portion of circuit and (2) in the surrounding magnetic and

electric fields. Then the effective resistance of the given portion of circuit is

$$r = \frac{P_h}{I^2} \tag{21}$$

This definition of effective resistance is applicable irrespective of the frequency and wave shape. The *value* of the effective resistance, however, does in general depend upon both these factors. When the magnetic field contains iron or other ferromagnetic substances the effective resistance also depends upon the strength of the current. However, in most practical cases, the variation in the value of the effective resistance with the current strength is comparatively slight. In the particular case of a linear conductor (*e.g.*, transmission line) or coil in which there is no skin effect or eddy currents or hysteresis, the effective resistance is of course equal to the ohmic, or direct-current resistance (see Article 174).

In Article 175 the impedance of a coil was defined as the ratio of its terminal voltage to the current through it. This definition may also be applied to any portion of an electric circuit, whether a coil or condenser, or to a combination of coils and condensers. That is, by the impedance z of any portion of an electric circuit is meant the factor by which the r.m.s. value I of the current in this circuit must be multiplied in order to give the r.m.s. value V_z of that portion of the voltage drop through it *due solely to its resistance, self-inductance and capacity*, viz.,

$$z = \frac{V_z}{I} \tag{22}$$

When there is in the given portion of the circuit an electromotive force of r.m.s. value E, due to mutual induction (as in a transformer), or to the cutting of lines of magnetic force (as in a motor or generator) the total voltage drop is not zI, but is the *vector sum* of zI and this "external" electromotive force, viz.,

$$V = \overline{E + zI} \tag{22a}$$

In this equation the positive sense of the electromotive force E is taken the same as the positive sense of the current I. When the positive sense of the electromotive force E is taken opposite to that of the current I, the terminal voltage V is the vector *difference* of E and the impedance drop zI, viz.,

$$V = \overline{E - zI} \tag{22b}$$

Compare with equation (6*a*) and (6) of Article 40 for a receiver and source of electric energy respectively. Also compare (22*b*) with the equation (40*a*) of Article 128 for the instantaneous value of the terminal voltage of an alternator.

The drop of potential V_z in any portion of an electric circuit due to its impedance is called the "impedance drop" in this portion of the circuit. *The impedance drop V_z is, as a rule, not in phase with the current.* However, the impedance drop may always be considered as made up of two components, one in phase with, and the other in quadrature with, the current. The product of the r.m.s. value of the current by the r.m.s. value of the active component of the impedance drop is always equal to the average rate at which heat is developed due to the

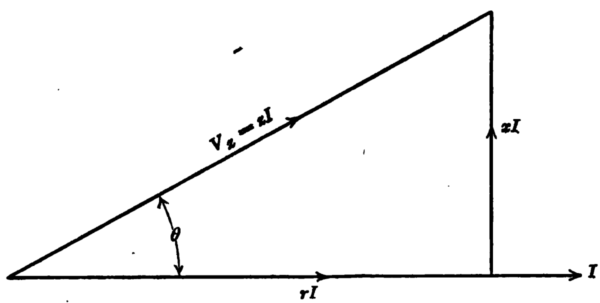


FIG. 137.—Impedance triangle.

effective resistance r of the portion of the circuit under consideration. Consequently, from equation (21), calling θ the angle by which the impedance drop *leads* the current (Fig. 137), then

$$(V_z \cos\theta)\, I = rI^2$$

or

$$V_z \cos\theta = rI \tag{23}$$

That is, the component of the impedance drop in phase with the current is always equal to the product of the current by the effective resistance of the portion of the circuit under consideration. This component is therefore equal to the effective resistance drop.

The other component of the impedance drop, namely, the component which *leads* the current by 90 degrees (see Fig. 137), is called the effective reactance drop.

The ratio of the r.m.s. value of this effective reactance drop to the r.m.s. value of the current is called the effective reactance. That is, calling x the effective reactance of any given portion of a circuit, the component of the impedance drop in quadrature with the current is

$$V_z \sin \theta = xI \tag{23a}$$

Combining equations (22), (23) and (23*a*), it is evident that the equivalent reactance may always be determined from the impedance z and effective resistance r by the formula

$$x = \sqrt{z^2 - r^2} \tag{24}$$

From the general definition given above, equation (23*a*), it is evident that the equivalent reactance of a given portion of a circuit may have either a positive or negative value, accordingly as the current lags or leads the impedance drop. When the given portion of the circuit is a coil of appreciable inductance, the current always lags the impedance drop. Consequently a coil has a positive reactance. When the given portion of circuit is a condenser, the current leads the voltage drop through it (see Fig. 136). Consequently a condenser has a *negative* reactance.

Problem 6.—Calculate from the test data given in Problem (5), Article 176, (*a*) the impedance, (*b*) the effective resistance, and (*c*) the effective reactance of the condenser there described.

Answer.—(*a*) 706 ohms. (*b*) 27 ohms. (*c*) −706 ohms.

179. Effective Resistance, Effective Reactance and Impedance of a Condenser.—Consider a condenser of capacity C and effective leakage conductance g. As shown in Article 176, the ratio of the total current I taken by such a condenser to the voltage V across its terminals, *i.e.*, the admittance of the condenser, is

$$y = \frac{I}{V} = \sqrt{g^2 + (2\pi fC)^2}$$

where f is the frequency of the impressed voltage. The impedance of this condenser, namely, the ratio of the impressed voltage to the total current taken by it, is therefore

$$z = \frac{V}{I} = \frac{1}{\sqrt{g^2 + (2\pi fC)^2}} \tag{25}$$

Also, as shown in Article 176, the heat dissipated in the condenser is gV^2. Consequently from equation (21), the effective resistance of the condenser is

$$r = \frac{gV^2}{I^2} = \frac{g}{g^2 + (2\pi fC)^2} \tag{25a}$$

Therefore, from equation (24), noting that the current *leads* the voltage drop (Fig. 136), the effective reactance of the condenser is

$$x = -\sqrt{z^2 - r^2} = \frac{-2\pi fC}{g^2 + (2\pi fC)^2} \tag{25b}$$

When the effective leakage conductance is negligible, as it usually is when the frequency is 60 cycles or less, the above expressions for the impedance, effective resistance and effective reactance of a condenser become respectively

$$z = \frac{1}{2\pi fC} \tag{25c}$$

$$r = 0 \tag{25d}$$

$$x = -\frac{1}{2\pi fC} \tag{25e}$$

180. Effective Conductance, Effective Susceptance and Admittance.—As shown in Article 177, the ohmic resistance, self-inductance and capacity of a given portion of a circuit, together with the heat losses in the magnetic and electric fields established by a current in it, may be conveniently expressed in terms of the three quantities—effective resistance, effective reactance, and impedance. This method of expression is always employed in any problem which has to do with two or more alternating-current circuits in *series* (see Article 181). However, in any problem involving two or more alternating-current circuits in parallel, it is usually more convenient to express these properties of each circuit in terms of the effective conductance, effective susceptance, and admittance of this circuit.

By the admittance y of any portion of a network, whether a coil or a condenser, or a combination of coils and condensers, is meant the reciprocal of its impedance, viz.,

$$y = \frac{1}{z}$$

The admittance is therefore equal to the current divided by the impedance drop. Or, when there is in the circuit no source of electromotive force other than that due to the self-inductance or capacity of the circuit, the admittance is equal to the total current in this circuit divided by the voltage between its terminals.

By the effective conductance g of any portion of a network is meant the factor by which the r.m.s. value V of the impedance drop in it must be multiplied in order to give the component of the total current in phase with this impedance drop. That is, calling θ the angle by which the total current lags the impedance drop, then

$$I \cos \theta = gV_z \tag{26}$$

Compare with equation (23).

Since, from Fig. 137, $\cos \theta = \frac{rI}{zI} = \frac{r}{z}$, and since $\frac{I}{V_z} = \frac{1}{z}$, it follows that the effective conductance is also

$$g = \frac{r}{z^2} = \frac{r}{r^2 + x^2}$$

where r is the effective resistance and x the effective reactance of the given portion of circuit.

Also, since the average power input to a circuit is equal to the product of the terminal voltage by the in-phase, or active, component of the current, it follows that when the total voltage drop V through the circuit is equal to the impedance drop V_z, then

$$g = \frac{P_h}{V^2} \tag{27}$$

where P_h is the average power input to the circuit, all of which appears as heat. Compare with equation (21), and also with equation (19*a*).

By the effective susceptance b of any portion of a network is meant the factor by which the r.m.s. value V_z of the impedance drop in it must be multiplied in order to give the component of the total current which *lags* this impedance drop by 90 degrees. That is, calling θ the angle by which the total current lags the impedance drop, then

$$I \sin \theta = bV_z \tag{28}$$

Compare with equation (43*a*).

Since, from Fig. 137, $\sin \theta = \frac{xI}{zI} = \frac{x}{z}$, and since $\frac{I}{V_z} = \frac{1}{z}$, it follows that the effective susceptance is also

$$b = \frac{x}{z^2} = \frac{x}{r^2 + x^2}$$

where r is the effective resistance and x the effective reactance of the given portion of the network.

Note that the effective susceptance, as thus defined, has the same algebraic sign as the effective reactance x. Consequently, the effective susceptance of a condenser is negative, viz., is equal to $-2\pi fC$, where C is the capacity of the condenser and f the frequency. To avoid confusion, therefore, the quantity $2\pi fC$ is called the *capacity* susceptance of the condenser, as distinguished from its *effective* susceptance $-2\pi fC$.

To sum up: when the impedance z, the effective resistance r, and the effective reactance x of any portion of a network are given, then the admittance y, the effective conductance g, and the effective susceptance b are respectively

$$y = \frac{1}{z} \tag{29}$$

$$g = \frac{r}{z^2} = \frac{r}{r^2 + x^2} \tag{29a}$$

$$b = \frac{x}{z^2} = \frac{x}{r^2 + x^2} \tag{29b}$$

From these relations it also follows that in terms of y, g and b, the values of z, r and x are respectively

$$z = \frac{1}{y} \tag{30}$$

$$r = \frac{g}{y^2} = \frac{g}{g^2 + b^2} \tag{30a}$$

$$x = \frac{b}{y^2} = \frac{b}{g^2 + b^2} \tag{30c}$$

Compare with equations (25) to (25*b*) for the special case of a condenser.

Note also that the angle by which the impedance drop *leads* the current (or by which the current lags the impedance drop) is

$$\theta = \tan^{-1}\left(\frac{x}{r}\right) = \tan^{-1}\left(\frac{b}{g}\right) \tag{31}$$

Also note that when the effective reactance is positive, the effective susceptance is also positive, and the power-factor angle θ is positive, *i.e.*, the impedance drop leads the current. When the effective reactance is negative, the effective susceptance is

also negative, and the power factor angle θ is negative, *i.e.*, the impedance drop *lags* the current.

Problem 7.—The armature winding of a single-phase alternator is short-circuited, and the machine is driven at normal speed, the field excitation being reduced so that the current produced is not sufficient to overheat the armature. The power required to drive the machine is 25 kilowatts, 5 kilowatts of which is expended in overcoming mechanical friction and windage. The armature current is 75 amperes, and the electromotive force generated in the armature winding due to its rotation in the magnetic field is 1250 volts.

(*a*) What is the relation between the electromotive force generated in the armature winding and the impedance drop in this winding? (*b*) What is the impedance of the armature winding? (*c*) Its effective resistance? (*d*) Its effective reactance? (*e*) Its admittance? (*f*) Its effective conductance? (*g*) Its effective susceptance? (*h*) What is the phase angle between the current and the impedance drop in the armature? (*i*) Draw to scale a vector diagram showing the components of the impedance drop in phase with, and in quadrature with, the current. (*j*) Draw to scale a vector diagram showing the components of the current in phase with, and in quadrature with, the impedance drop.

Answer.—(*a*) The impedance drop is equal to and in phase with the generated electromotive force, for, since the terminal voltage is zero, the *rise* of potential in the direction of the current due to the generated electromotive force must be at each instant equal to the *drop* of potential in this same direction due to the impedance of the armature. (*b*) 16.6 ohm. (*c*) 3.55 ohm. (*d*) 16.2 ohm. (*e*) 0.06 mho. (*f*) 0.0128 mho. (*g*) 0.0586 mho. (*h*) 77.5 degrees.

181. Impedances in Series.—Any portion of an alternating-current circuit or network which possesses the property of resistance, inductance or capacity may be conveniently referred to as *an* impedance, just as a portion of a direct-current network which possesses the property of resistance is called *a* resistance. Two or more impedances, *i.e.*, two or more portions of a circuit or network, are said to be in series when the same alternating current flows through each. Similarly, two or more impedances are said to be in parallel when the same voltage is established across the terminals of each.

The relations given below are deduced on the assumption that all currents and voltages in the circuit or network are sinusoidal, *i.e.*, that the wave forms of the currents and voltages are sine waves. When the currents and voltages are non-sinusoidal, but alternate between equal positive and negative maximum values, these relations, though not strictly applicable, will, in

most practical cases, give sufficiently accurate results, provided the non-sinusoidal currents and voltages are considered as equivalent to sine-wave currents and voltages of the same r.m.s. values, and differing in phase by the equivalent phase difference, as defined in Article 170.

Consider two or more impedances z_1, z_2, etc., in series (Fig. 138), and let r_1, r_2, etc., be the effective resistances of the several im-

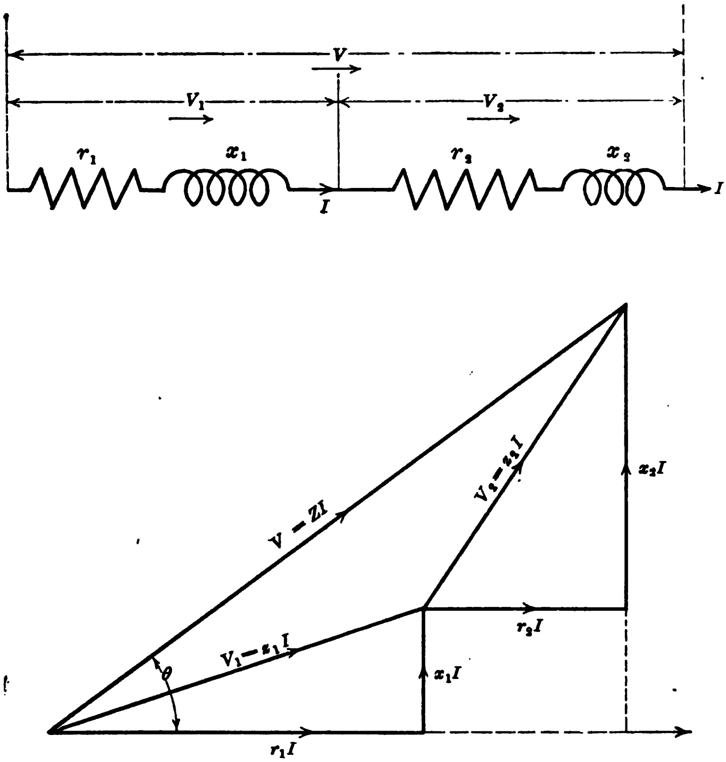


FIG. 138.—Impedances in series.

pedances and x_1, x_2, etc., the effective reactances of these impedances. Let I be the r.m.s. value of the current in each impedance. Then, since the current in each impedance is the same, the resistance drops r_1I, r_2I, etc., in the several impedances are all in phase with each other, since they are all *in phase with the current.* Also, the reactance drops x_1I, x_2I, etc., in the several impedances are all in phase with each other, since each *leads the current by* 90 *degrees.*

Hence the several impedances in series are equivalent to a single impedance Z of such a resistance R that the resistance drop in this impedance is

$$RI = r_1I + r_2I + \text{etc.}$$

and of such a reactance X that the reactance drop in this impedance is

$$XI = x_1I + x_2I + \text{etc.}$$

Whence, dividing each of these expressions by I, the several impedances in series are equivalent to a single impedance Z whose resistance R is the arithmetical sum of the resistances of the several impedances, viz.,

$$R = r_1 + r_2 + \text{etc.} \tag{32}$$

and whose reactance X is the algebraic sum of the reactances of the several impedances, viz.,

$$X = x_1 + x_2 + \text{etc.} \tag{32a}$$

Note that equation (32) is an arithmetical sum, since resistances are always positive. On the other hand, since a reactance may be positive or negative (*i.e.*, inductive or condensive), equation (32*a*) is an algebraic summation.

The value of the impedance Z equivalent to the several impedances in series is then

$$Z = \sqrt{R^2 + X^2} \tag{33}$$

where R and X are given by (32) and (32*a*) respectively. The power-factor angle of this impedanee is

$$\theta = \tan^{-1}\frac{X}{R} \tag{33a}$$

From the relations just deduced it is evident that impedances in series in an alternating-current circuit cannot be treated as resistances in series in a direct-current circuit. **Impedances can be combined only by resolving them into their resistances and reactances, they MUST NOT be added arithmetically.**

The effect of an impedance in series with a load of any kind, whether a simple impedance or a transformer or motor, may also be determined from a vector diagram of the type shown in Fig. 138. For example, suppose it is desired to find the voltage V_o at the sending end of single-phase transmission line (see Fig. 139) required to maintain a voltage V at the terminals of a load con-

nected to the other end of the line, when the current taken by this load is I amperes and lags the voltage drop through the load by the angle θ. Let r be the total resistance and x the total reactance of the line (both wires). Then, since the resistance drop rI in the line is in phase with the current, and the reactance drop in the line leads the current by 90 degrees, the problem reduces to the solution of the vector diagram shown in the figure.

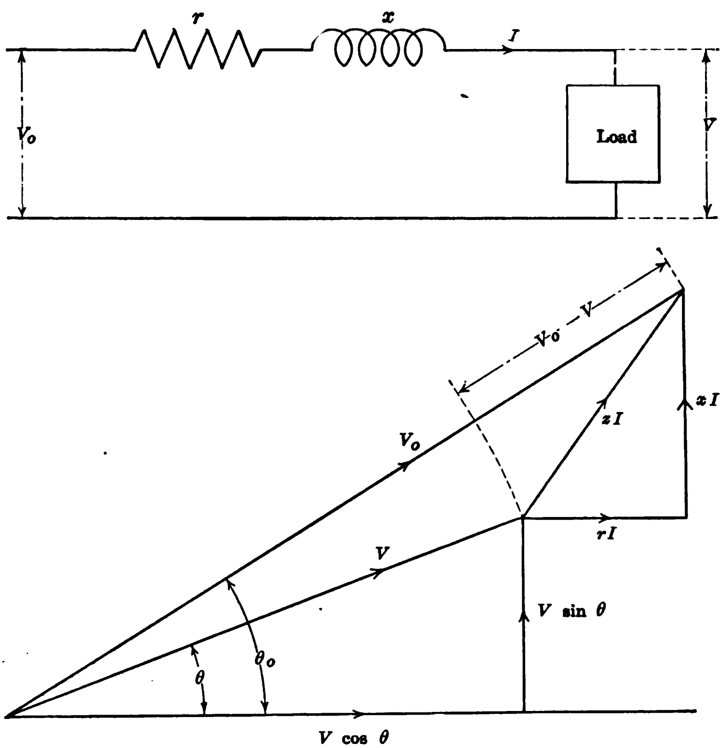


FIG. 139.—Transmission line and load.

From the diagram it is evident that the voltage at the sending end of the line is

$$V_o = \sqrt{(V \cos \theta + rI)^2 + (V \sin \theta + xI)^2} \tag{34}$$

and that the current lags this voltage by the angle

$$\theta_o = \tan^{-1}\left[\frac{V \sin \theta + xI}{V \cos \theta + rI}\right] \tag{34a}$$

Numerical calculations by means of these formulas are greatly simplified by putting

$$A = \frac{rI}{V} = \frac{\text{resistance drop in line}}{\text{load voltage}} \tag{35}$$

$$B = \frac{xI}{V} = \frac{\text{reactance drop in line}}{\text{load voltage}} \tag{35a}$$

Then (34) and (34*a*) reduce to

$$V_o = V\sqrt{(\cos\theta + A)^2 + (\sin\theta + B)^2} \tag{36}$$

$$\theta_o = \tan^{-1}\left(\frac{\sin\theta + B}{\cos\theta + A}\right) \tag{36a}$$

It should be carefully noted that the difference between the numerical values of the sending end and receiving end voltages, namely, the algebraic difference ($V_o - V$), is not equal to the impedance drop zI in the transmission line. This is evident from Fig. 139. The difference between the numerical values of the sending and receiving end voltages is usually referred to as the voltage lost in the line. Hence the voltage lost in an alternating-current transmission line is NOT equal to the impedance drop in the line.

Fig. 139 and the equations just deduced are also applicable to the calculation of the r.m.s. value and phase angle θ_o of the generated voltage V_o (usually represented by the symbol E) in terms of its terminal voltage V, current output I, power factor angle θ of the load supplied by an alternator, and the internal resistance r and internal reactance x of the armature winding; see also equation (22*b*), Article 178. It must not be forgotten, however, that this diagram and the equations deduced from it are all on the assumption of constant self-inductance and sine-wave currents and voltages. The assumption of constant internal self-inductance is only a rough approximation in the case of an alternator.

Problem 8.—A coil which has a resistance of 5 ohms and a reactance of 2 ohms is connected in series with a coil whose resistance is 1 ohm and whose reactance is 12 ohms.

(*a*) What is the resistance and reactance of the single impedance equivalent to these two coils? (*b*) What is the numerical value of each of the two impedances? (*c*) What is the numerical value of the single impedance equivalent to these two? (*d*) What is the power factor angle of this equivalent impedance. (*e*) Draw to scale the vector diagram for this circuit when the current is 1 ampere.

Answer.—(*a*) 6 ohms and 14 ohms. (*b*) 5.38 and 12.04 ohms. (*c*) 15.25 ohm. (*d*) 66.8 degrees.

Problem 9.—A transmission line 25 miles long consists of two No. 0 A. W. G. stranded copper wires, spaced 3 feet between centers. This line supplies a 60-cycle load of 1500 kilowatts at 25,000 volts. The power factor of the load is 75 per cent., the current lagging the voltage. The leakage and electrostatic capacity of the line may be neglected.

(*a*) What is the voltage at the power station (sending) end? (*b*) What is the total load supplied to the line at the power station end? (*c*) What is the power factor of this total load? (*d*) What would be the voltage at the power station were the power factor at the load end of the line 75 per cent. as before, but the current *leading*, instead of lagging, the load voltage? (*e*) Under these conditions, what would be the power factor of the total load supplied to the line at the power station end? (*f*) Draw to scale vector diagrams for both cases, *i.e.*, for the current lagging and for the current leading.

Answer.—(*a*) 28,400 volts. (*b*) 1673 kilowatts. (*c*) 73.7 per cent. (*d*) 25,000 volts. (*e*) 83.4 per cent.

Problem 10.—The terminal voltage of a certain single-phase 60-cycle generator when delivering a current of 100 amperes at 90 per cent. power-factor lagging is 2200 volts. The armature winding has an effective resistance of 0.5 ohm and an effective reactance of 3 ohms (assumed constant). Assume sine-wave voltages and currents.

(*a*) What is the r.m.s. value of the electromotive force generated in the armature winding due to its rotation? (*b*) What is the difference in phase between this electromotive force and the armature current? (*c*) What is the maximum value of this generated electromotive force. (*d*) If the field current is 5 amperes, what is the maximum value of the mutual inductance between the field winding and armature winding.

Answer.—(*a*) 2390 volts. (*b*) 31.8 degrees. (*c*) 3380 volts. (*d*) 1.79 henries.

182. Coil and Condenser in Series. Resonance.—A special case of two impedances in series, which is of particular interest, is that of a coil in series with a condenser (Fig. 140). Let r be the resistance and L the self-inductance of the coil, let C be the capacity of the condenser, and let f be the frequency of the impressed voltage. The conductance of the condenser will be neglected. The inductance L and the capacity C are both assumed constant, and the current established in the circuit is assumed to have a sine-wave form.

Under these conditions the reactance of the coil will be $2\pi fL$. The reactance of the condenser will be $-\frac{1}{2\pi fC}$, since the voltage drop through the condenser *lags* the current taken by it (see

Article 179). Since the conductance of the condenser is zero (by hypothesis), its effective resistance is likewise zero.

Hence, when the current established is a sine-wave current, the coil and condenser in series are equivalent to a single impedance whose resistance is equal to the resistance r of the coil, and whose reactance is

$$X = 2\pi fL - \frac{1}{2\pi fC} \tag{37}$$

That is, the reactance of a coil and condenser in series is the *difference* between the *inductive* reactance of the coil and the *capacity* reactance of the condenser.

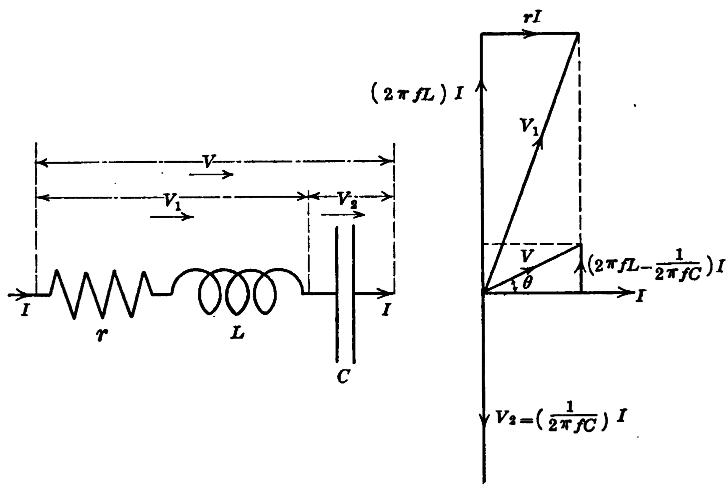


FIG. 140.—Coil and condenser in series.

The single impedance equivalent to the coil and condenser in series therefore has the value

$$Z = \sqrt{r^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2} \tag{37a}$$

The sine-wave current established in such a circuit by a sine-wave voltage of r.m.s. value V will therefore have an r.m.s. value

$$I = \frac{V}{Z} \tag{38}$$

where Z is given by equation (37a). The angle by which the current lags the impressed voltage is

$$\theta = \tan^{-1}\left(\frac{X}{r}\right) \tag{38a}$$

where X is given by equation (37).

These relations are all clearly shown by the vector diagram in Fig. 140. Note particularly the phase relations and relative magnitudes of the resistance drop in the coil, the reactance drop in the coil, and the voltage across the condenser.

When the inductance L and capacity C are of such values that

$$2\pi fL = \frac{1}{2\pi fC}$$

or when

$$f = \frac{1}{2\pi\sqrt{LC}} \tag{39}$$

the equivalent reactance X is zero. The current in the circuit will then have its *maximum* r.m.s. value, namely,

$$I_{max.} = \frac{V}{r} \tag{40}$$

That is, when the condition expressed by equation (39) holds, the current in a coil and a condenser in series is exactly the same as would be established by the given sine-wave impressed voltage in a *non-inductive* resistance equal to that of the coil. The power factor of the circuit under these conditions is unity. A condenser in series with a coil may therefore be used to neutralize the effect of the self-inductance of the coil. The circuit is then said to be "in resonance" with the impressed voltage.

The frequency f which gives maximum current in the circuit, for a given impressed voltage, is the natural frequency of the coil and condenser (see Article 158). Note the analogy with the motion of a body, which is free to vibrate, produced by a periodic force having a period equal to the free period of vibration of the body. For example, a heavy church bell may be caused to swing with large amplitude (corresponding to the maximum value of the current I) when a comparatively small force (corresponding to the voltage V) is applied by the man pulling the rope, provided the successive applications of this force are in time with the free swings of the bell.

It should be particularly noted that, although the resultant voltage across the resistance, inductance and capacity in series, is equal to rI, *i.e.*, is the same as the voltage across the resistance, the voltage across the inductance or across the condenser may be many times this. For example, when the inductance L is 1

henry and the capacity C is 7.04 microfarads, and the frequency is 60 cycles, the reactance of the coil $2\pi \times 60 \times 1 = 377$ ohms, and the capacity reactance of the condenser is

$$x_c = \frac{1}{2\pi \times 60 \times 7.04 \times 10^{-6}} = 377 \text{ ohms.}$$

Hence the total reactance of the circuit is zero, and the circuit is therefore in resonance with the impressed voltage.

When the resistance r is 1 ohm and the impressed voltage across the entire circuit is 100 volts, the current is

$$I = \frac{100}{\sqrt{(1)^2 + (0)^2}} = 100 \text{ amperes}$$

The resistance drop in the coil is then $100 \times 1 = 100$ volts and the inductive drop in the coil is $377 \times 100 = 37{,}700$ volts. The resultant voltage across the coil is then $\sqrt{(100)^2 + (37{,}700)^2} = 37{,}700$ volts. The voltage across the condenser is likewise $377 \times 100 = 37{,}700$ volts. This brings out in a striking manner the fundamental fact that *alternating voltages cannot be added algebraically; they must be added vectorially.*

The relations above deduced are all on the assumption that the inductance of the coil is a constant (no iron core) and that the wave-form of the impressed voltage is a sine-wave. When the coil has an iron core, or when the impressed voltage is not sinusoidal, there will still be a certain frequency which will give maximum current, but the power factor of the circuit will not be unity. This is due to the fact that the reactance of the circuit, for a given value of the inductance and capacity, can be zero only to the fundamental, or to one of the harmonics, in the impressed voltage (see Article 213).

It should also be noted that the relations above deduced do not apply to the transient effects produced in such a circuit (see Article 158) immediately after a voltage is impressed.

Problem 11.—A coil which has a resistance of 0.1 ohm and a constant inductance of 0.3 henry is connected in series with a condenser whose capacity is 23.5 microfarads. A voltage in excess of 2000 volts will puncture the condenser.

(*a*) What is the natural frequency of this circuit? (*b*) What is the r.m.s. value of the maximum 60-cycle, sine-wave voltage which may be impressed on the terminals of this circuit without rupturing the dielectric of the condenser? (*c*) What will be the current in the circuit under these

conditions? (*d*) What will be the voltage across the terminals of the coil under these conditions? (*e*) What is the r.m.s. value of the maximum 25-cycle, sine-wave voltage which could be impressed on the circuit without puncturing the condenser? (*f*) What would be the current in the circuit under these conditions? (*g*) What would be the voltage across the terminals of the coil? (*h*) Draw to scale a complete vector diagram for both (*b*) and (*e*).

Answer.—**(*a*) 60 cycles per second. (*b*) 1.77 volts. (*c*) 17.7 amperes. (*d*) 2000 volts. (*e*) 1653 volts. (*f*) 7.38 amperes. (*g*) 347 volts.**

Problem 12.—A sine-wave voltage of constant r.m.s. value of 1 volt is impressed on the circuit described in Problem 10, and the frequency of this voltage is varied (by changing the speed of the generator) from 50 cycles per second to 70 cycles per second. Plot to scale the current in the circuit against frequency as abscissas. Such a curve is called a "resonance curve."

183. Impedance in Parallel.—Just as two or more impedances in series are equivalent to a single impedance of definite resistance and reactance, so may two or more impedances connected in parallel be replaced by a single impedance, provided (1) that the resistance, self-inductance and capacity of each impedance is constant, (2) that the impressed voltage and the currents are sinusoidal, and (3) that there is no mutually induced electromotive force, or electromotive force due to mechanical motion, in any one of these impedances. The value of this equivalent impedance will now be deduced.

Referring to Fig. 141, let r_1, r_2, etc., be the effective resistances and x_1, x_2, etc. be the effective reactances of two or more impedances which are connected in parallel. The effective conductances and susceptances of these impedances are then (see Article 180).

$$g_1 = \frac{r_1}{r_1^2 + x_1^2}, \qquad b_1 = \frac{x_1}{r_1^2 + x_1^2}$$

$$g_2 = \frac{r_2}{r_2^2 + x_2^2}, \qquad b_2 = \frac{x^2}{r_2^2 + x_2^2}$$

Let V be the r.m.s. value of the voltage across each impedance.

The current in each branch may be considered as made up of two components, one in phase with the voltage drop through it, and the other lagging this voltage drop by 90 degrees. From equations (26) and (28), the components in phase with the voltage drop are respectively g_1V, g_2V, etc., and the components which lag the voltage drop by 90 degrees are respectively b_1V, b_2V, etc.

Since the voltage drop is the same in each impedance, both in magnitude and phase, the total current through the several impedances has a component in phase with the voltage drop equal to

$$(g_1 + g_2 + \text{etc.})V$$

and a component lagging this voltage drop by 90 degrees equal to

$$(b_1 + b_2 + \text{etc.})V$$

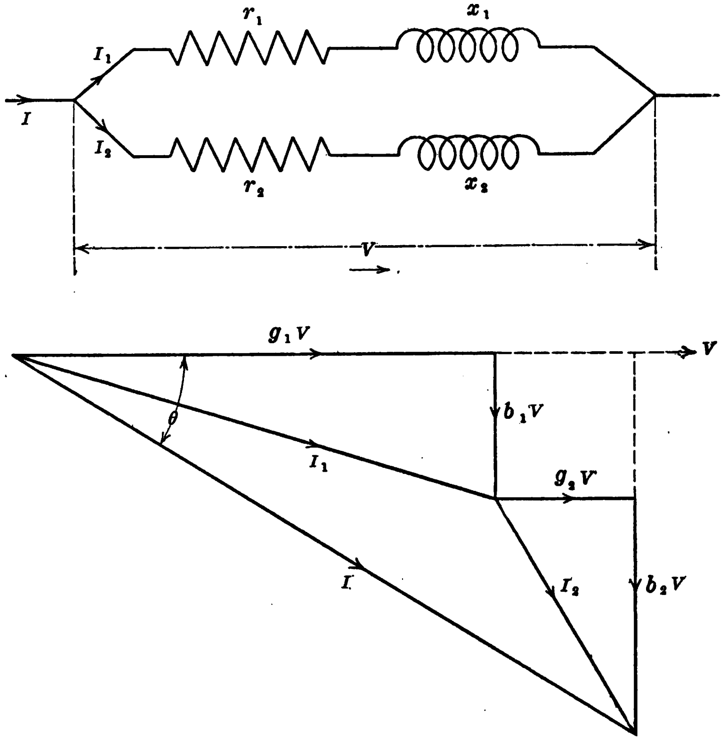


FIG. 141.—Impedances in parallel.

These two components are exactly the same as would be produced by the same voltage V in a single impedance Z whose conductance is

$$G = g_1 + g_2 + \text{ etc.} \tag{41}$$

and whose susceptance is

$$B = b_1 + b_2 + \text{ etc.} \tag{41a}$$

The admittance corresponding to this equivalent impedance is

$$Y = \sqrt{G^2 + B^2} \tag{41b}$$

The numerical value of this equivalent impedance is then

$$Z = \frac{1}{Y} \tag{42}$$

and its power factor angle is

$$\theta = \tan^{-1}\left(\frac{B}{G}\right) \tag{42a}$$

The resistance and reactance of the equivalent impedance Z are

$$R = Z \cos \theta = \frac{G}{G^2 + B^2} \tag{43}$$

$$X = Z \sin \theta = \frac{B}{G^2 + B^2} \tag{43a}$$

In calculating the equivalent susceptance B, it must be remembered that a capacity susceptance is to be treated as *negative* quantity. Hence equation (41*a*) represents an *algebraic* summation.

Problem 13.—A coil which has a resistance of 5 ohms and a reactance of 2 ohms is connected in parallel with a coil whose resistance is 1 ohm and whose reactance is 12 ohms.

(*a*) What is the effective conductance and susceptance of the single impedance equivalent to these two coils? (*b*) What is the admittance of this equivalent impedance. (*c*) What is the numerical value of this equivalent impedance? (*d*) What is its power-factor angle? (*e*) What is the resistance and reactance of this equivalent impedance? (*f*) Draw to scale the vector diagram for this circuit when the impressed voltage is 1 volt. Compare with Problem 8, Article 181.

Answer.—(*a*) 0.1790 mho and 0.1524 mho. (*b*) 0.235 mho. (*c*) 4.25 ohms. (*d*) 40.4 degrees. (*e*) 3.27 ohms and 2.75 ohms.

184. Coil and Condenser in Parallel. Resonance.—A coil and condenser in parallel (see Fig. 142), like a coil and condenser in series, is a special case of two impedances of particular interest. Let r be the resistance and L the self-inductance of the coil, let C be the capacity of the condenser, and let f be the frequency of the impressed voltage. The conductance of the condenser will be neglected, and the resistance of the coil will be assumed small in comparison with its reactance. The inductance L and capacity C are both assumed constant, and the currents established are assumed to be sinusoidal.

Under these conditions the reactance of the coil is $2\pi fL$. Hence its effective conductance and susceptance are respectively

$$g_1 = \frac{r}{r^2 + (2\pi fL)^2} \qquad \text{and} \qquad b_1 = \frac{2\pi fL}{r^2 + (2\pi fL)^2}$$

since, by hypothesis, r is small in comparison with $(2\pi fL)$, these expressions may, to a close approximation, be written

$$g_1 = \frac{r}{(2\pi fL)^2} \qquad \text{and} \qquad b_1 = \frac{1}{2\pi fL}$$

The conductance and susceptance of the condenser are respectively

$$g_2 = 0 \qquad \text{and} \qquad b_2 = -\,2\pi fC$$

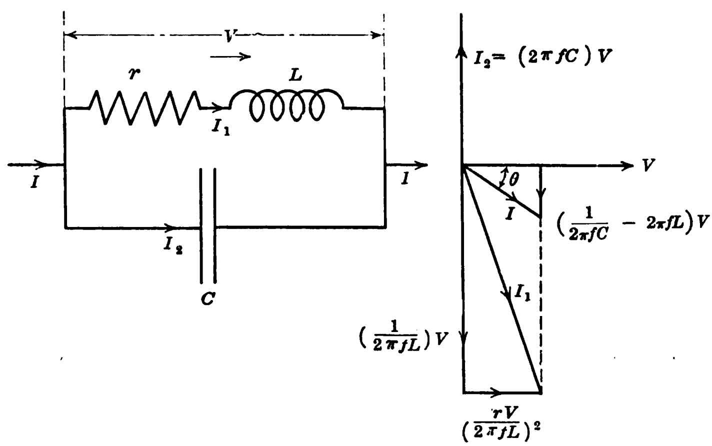


FIG. 142.—Coil and condenser in parallel.

Hence, when the currents established are sine-wave currents, the coil and condenser in parallel are equivalent to a single impedance whose conductance is

$$G = \frac{r}{(2\pi fL)^2} \tag{44}$$

and whose susceptance is

$$B = \frac{1}{2\pi fL} - 2\pi fC \tag{44a}$$

Compare with equation (37) for a coil and condenser in series. The coil and condenser in parallel is then equivalent to a single admittance whose value is

$$Y = \sqrt{\left[\frac{r}{(2\pi fL)^2}\right]^2 + \left(\frac{1}{2\pi fL} - 2\pi fC\right)^2} \qquad (44b)$$

The r.m.s. value of the resultant current taken by such a circuit when a sine-wave voltage of r.m.s. value V is impressed is therefore

$$I = YV \qquad (45)$$

where Y is given by equation (44*b*). The angle by which this resultant current lags the impressed voltage is

$$\theta = \tan^{-1}\left(\frac{B}{G}\right) \qquad (46)$$

where G and B are given by equations (44) and (44*a*). These relations are all clearly shown by the vector diagram in Fig. 142.

When the inductance L and capacity C are of such values that the natural frequency of the coil and condenser is equal to the frequency of the impressed voltage, namely, when

$$f = \frac{1}{2\pi\sqrt{LC}} \qquad (47)$$

the equivalent susceptance B is zero, just as the equivalent reactance of the coil and condenser in series is zero (see Article 182). Under these conditions the resultant current has its *minimum* value, namely,

$$I_{min} = \frac{rV}{(2\pi fL)^2} \qquad (48)$$

and is in phase with the impressed voltage. That is, when a coil and a condenser of constant resistance, inductance and capacity are connected in parallel, and are in resonance with a sine-wave impressed voltage, the current which is supplied to this circuit is a *minimum*, and the power factor is unity.

It should be particularly noted, however, that although the resultant current is a minimum, the currents in the coil and condenser may be many times greater than this resultant current. For example, when the inductance L is 1 henry and the capacity C is 7.04 microfarads, and the frequency is 60 cycles per second, the susceptance b_1 of the coil is 0.00265 mho and the susceptance b_2 of the condenser is -0.00265 mho. The resultant susceptance B is then zero, and the circuit is therefore in resonance with the impressed voltage. When the resistance of the coil is

1 ohm, the effective conductance g, is 0.00000704 mho. Neglecting the conductance of the condenser, the total conductance G has this same value.

When the voltage impressed on the circuit is 100 volts, the resultant current is then

$$I = 100\sqrt{(0.00000704)^2 + (0)^2} = 0.000704 \text{ ampere.}$$

The active, or in-phase, component of the current in the coil is $0.00000704 \times 100 = 0.000704$ ampere, and the reactive component of the current in the coil is $0.00265 \times 100 = 0.265$ ampere. The resultant current in the coil is then

$$\sqrt{(0.000704)^2 + (0.265)^2} = 0.265 \text{ ampere.}$$

The current taken by the condenser (*i.e.*, its charging current) is likewise $0.00265 \times 100 = 0.265$ ampere. The currents in the coil and condenser are therefore each 377 times the resultant current. Compare with the relation between the voltages across the coil and condenser *in series* and the resultant or impressed voltage (see Article 182).

For the reasons stated in the discussion of a coil and condenser in series, the relations just deduced do not apply when the coil has an iron core, or when the impressed voltage is not sinusoidal, nor does it apply to the transient effects set up immediately after the voltage is impressed.

Problem 14.—A coil which has a resistance of 0.1 ohm and a constant self-inductance of 0.3 henry is connected in parallel with a condenser whose capacity is 23.5 microfarads. A sine-wave voltage of 1000 volts is impressed on this circuit, and the frequency is varied from 50 cycles per second to 70 cycles per second. Plot to scale the current in the circuit against frequency as abscissas. Compare with Problem 11, Article 182.

Answer.—One point on curve: $f = 55$ cycles per second, current = 3.04 amperes.

XVI

POLYPHASE CIRCUITS

185. Introduction.—A polyphase generator is an alternator which has two or more independent armature windings, arranged with respect to each other in such a manner that the electromotive forces in the several windings differ in phase by a constant angle.

The group of windings and conductors which form the paths of the currents produced by such electromotive forces is called

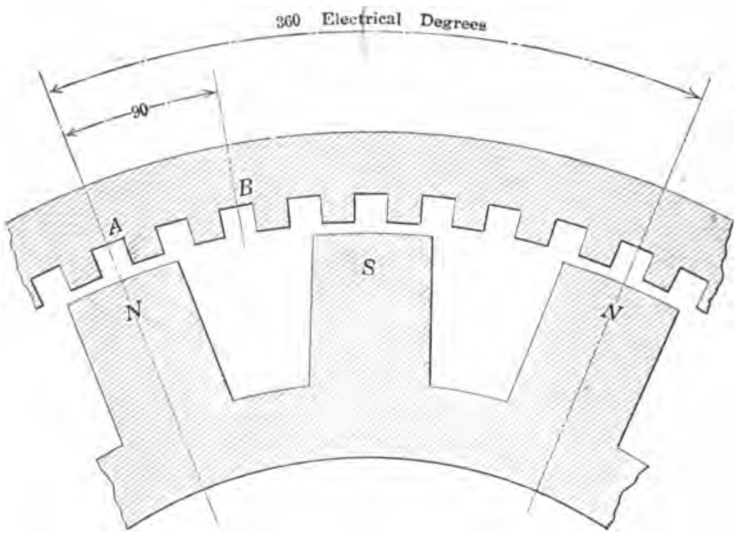


FIG. 143.—Two-phase alternator.

a polyphase circuit or system. The polyphase systems commonly employed are the two-phase and three-phase systems.

A two-phase generator is an alternator which has two independent armature windings, exactly alike, and so arranged on the armature core that when a given conductor of one winding is directly under the center of a north pole, for example, in the slot *A* in Fig. 143, the corresponding conductor in the other winding is midway between a north and south pole, *e.g.*, in the slot marked *B*

in Fig. 143. In the case of a two-pole machine this means that corresponding conductors in the two windings are 90 degrees apart, as measured around the circumference of the armature. In the case of a multipolar machine the corresponding conductors in the two windings are said to be 90 "electrical" degrees apart. By an electrical degree is here meant $\frac{1}{360}$th of the arc included between axes of two field poles of *like* sign, as indicated in Fig. 143.

A three-phase generator is an alternator which has three independent amature windings, exactly alike, and so arranged that when a given conductor of one winding is directly under the center

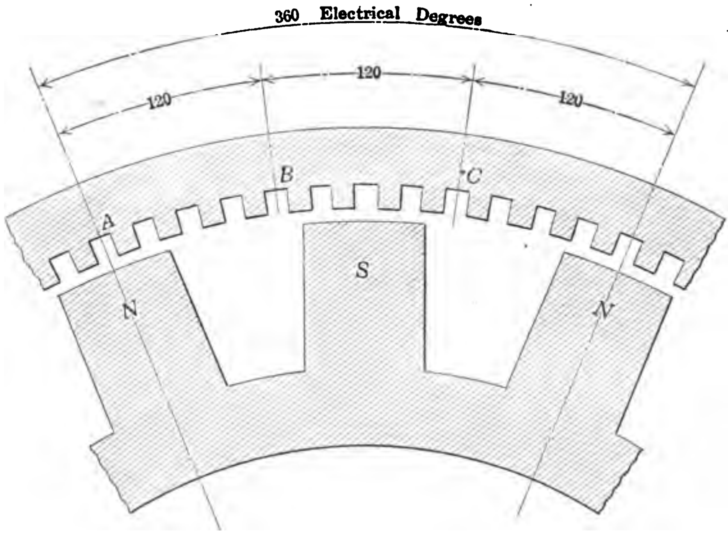


FIG. 144.—Three-phase alternator.

of a north pole, for example, in the slot *A* in Fig. 144, the corresponding conductor in the second winding is 120 electrical degrees from it, *e.g.*, in the slot marked *B* in Fig. 144, and the corresponding conductor in the third winding is 120 electrical degrees further around the armature, *e.g.*, in the slot marked *C* in Fig. 144.

In both two-phase and three-phase generators, except small laboratory machines, the rotating member carries the field winding, and the armature windings and core are stationary, as indicated in Figs. 143 and 144. The field winding is supplied with direct current, usually from a small direct-current generator, called an exciter. This exciting current enters and leaves the

field windings through slip rings mounted on the shaft of the machine.

Since the two windings of a polyphase machine are alike in number and arrangement of conductors, the electromotive forces generated in them by the relative motion of the field and armature are equal in r.m.s. value. The differences in phase between these electromotive forces is equal to the number of electrical degrees between corresponding conductors in the respective windings.

For example, the electromotive forces generated in the two windings of a two-phase machine differ in phase by 90 degrees. An inspection of Fig. 143 will make this evident. When the conductor in slot *A* is cutting lines of force at the maximum rate, the corresponding conductor of the other winding, located in slot *B*, is cutting no lines of force. Hence, when the electromotive force in the conductor in slot *A* is a maximum, the electromotive force in the conductor in slot *B* is zero. Similarly, the electromotive forces generated in the three windings of a three-phase machine differ in phase by 120 degrees.

Practically all large alternators are either three-phase or two-phase machines; usually three-phase. All long-distance alternating-current transmission lines are three-phase. Wherever alternating-currents are used for motors (other than alternating-current series motors) the three-phase or two-phase system is employed. Probably over 95 per cent. of all the electric energy used commercially is generated either three-phase or two-phase (most of it three-phase). Even when direct-current energy's desired, it is usually more economical to generate it as three-phase alternating-current energy, and then convert this alternating-current energy into direct-current energy.

The chief advantages of polyphase over single-phase systems are:

(*a*) A polyphase generator is cheaper than a single-phase generator of the same output, voltage, and speed.

(*b*) A polyphase generator has a higher efficiency and better voltage regulation than a single-phase generator of the same general design and rating.

(*c*) A three-phase transmission line, for the same voltage between wires, requires only 75 per cent. of the copper required

for a single-phase line in order to transmit the same amount of power over the same distance at the same efficiency.

(*d*) Polyphase motors are cheaper, and also have better operating characteristics, than single-phase motors.

186. Phase Currents and Line Currents. Phase Voltages and Line Voltages.—Balanced Currents, Voltages and Loads.— Each of the windings of a polyphase generator or motor, or any branch of any kind of a polyphase load, is commonly referred to as a "phase."

As a rule, the several phases of a polyphase machine are permanently connected so that they form a "star" or "mesh" (see Figs. 148 and 149), and the terminals of the machine are the free ends of the star, or the junction points of the mesh. When a given terminal is a junction point between two phases, the current which enters or leaves the machine at this terminal is the *resultant* of the currents in the two windings connected to it. The current which enters or leaves a machine at any given terminal is called the "line current," and the current in any particular winding or phase, is called the "phase current."

Similarly, the voltage between any two terminals is called the "line voltage," and the voltage across any particular winding or phase, is called the "phase voltage."

When the several phase currents in a polyphase machine are equal in r.m.s. value and differ successively in phase by the same angle, these currents are said to be "balanced." Similarly, when the several phase voltages are equal in r.m.s. value and differ successively in phase by the same angle, these voltages are said to be balanced. When both the currents and voltages are balanced the power input to, or output from, each of the phases is the same, and is equal to the total input to, or output of, the machine divided by the number of phases. The machine is then said to carry a "balanced load."

187. Double Subscript Notation.—When the current flowing through a conductor from any point 1 to any point 2 has at any particular instant the value

$$i_{12} = \sqrt{2}\ \mathrm{I} \sin \omega t$$

this current may always be considered as equivalent to a current

$$i_{21} = -\sqrt{2}\ I \sin \omega t$$

flowing through the conductor in the direction from 2 to 1. Hence if I_{12} is used to designate the vector which represents the current from 1 to 2, then the current from 2 to 1 may be represented by a vector I_{21} equal in length to I_{12} but drawn in the opposite direction, viz.,

$$I_{21} = -I_{12} \tag{1}$$

This use of double subscripts to designate the direction in space of a sine-wave quantity represented by a given vector will be found extremely convenient in the discussion of polyphase circuits, and the student will avoid much confusion by so marking all vectors in any diagram which he may construct.

As noted in Article 178, an electromotive force produced in an alternating-current circuit as the result of the motion of this circuit in a magnetic field, or as the result of a varying current in another circuit, may be represented by the symbol E. (The electromotive forces due to self-induction and capacity are taken into account by the reactance of the circuit.) An electromotive force is always taken positive in the direction corresponding to a *rise* of electric potential (see Article 41). Consequently, when an electromotive force is represented by a vector designated E_{12}, this vector represents the *rise* of potential from 1 to 2 due to the given electromotive force.

As in the preceding articles, the terminal voltage of a winding will be represented by the symbol V, and its positive sense will always be taken in the direction of the *drop* of potential. That is, when a terminal voltage is represented by a vector V_{12}, this vector represents the resultant *drop* of potential from 1 to 2 due to whatever electromotive forces or impedances there may be in the path under consideration.

For example, when the electromotive force and current in a winding of impedance Z are designated E_{12} and I_{12}, then the resultant *rise* of potential from 1 to 2 is the *vector difference* $\overline{E_{12} - zI_{12}}$. Since a drop is equivalent to a *negative* rise, the resultant *drop* of potential from 1 to 2 is then

$$V_{12} = \overline{-E_{12} + zI_{12}} \tag{2}$$

Or, the drop of potential from 2 to 1 (equal to $-V_{12}$ or V_{21}) is

$$V_{21} = \overline{E_{12} - zI_{12}} \tag{2a}$$

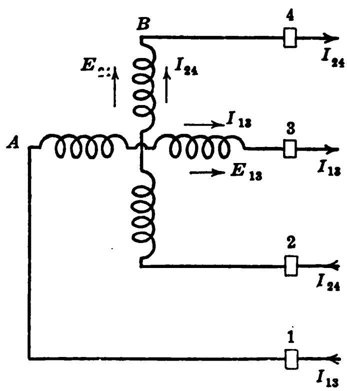
When the positive senses of the electromotive force and current

in a given winding are taken in *opposite* directions, *i.e.*, when the current is designated I_{12} and the electromotive force E_{21}, then, the drop of potential from 1 to 2 is

$$V_{12} = \overline{E_{21} + zI_{12}} \tag{2b}$$

In these expressions the line over the second member indicates vector addition or subtraction.

Equation (2*a*) is the more convenient expression for the terminal voltage of a source of alternating-current energy, and equation (2*b*) is the more convenient expression for the terminal voltage of a receiver of alternating-current energy. Compare with equations (22*b*) and (22*a*) of Article 178, and (6) and (6*a*) of Article 40.



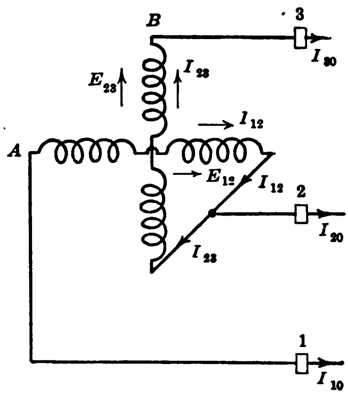


Fig. 145*A*. Fig. 145*B*.

Four-wire and three-wire two-phase alternator.

In the above discussion the subscripts 1 and 2 are used in a general sense, viz., the terminals may be designated 3 and 4, or *a* and *b*, etc., not necessarily 1 and 2.

188. Two-phase Systems.—As noted in Article 185, a two-phase machine is one whose armature has two independent windings, so arranged that the electromotive forces generated in them have the same r.m.s. value and differ in phase by 90 degrees.

The two terminals of the two windings, or "phases," may each be brought out to separate terminal blocks, as indicated in Fig. 145*A*, or one terminal of one phase may be permanently connected inside the machine to one of the terminals of the other phase, as indicated in Fig. 145*B*. In the first case, the machine is

called a "four-wire" two-phase machine; in the second case it is called a "three-wire" two-phase machine.

The electric power output (or input) of a two-phase machine is equal to the sum of the power outputs (or inputs) of the two armature windings. That is, calling V_a and V_b the r.m.s. values of the terminal voltages of the two windings, I_a and I_b the r.m.s. values of the currents in these windings, and θ_a the angle by which I_a lags V_a, and θ_b the angle by which I_b lags V_b, then the total output (or input) is

$$P = V_a I_a \cos \theta_a + V_b I_b \cos \theta_b \qquad (3)$$

When the terminal voltages of the two windings and the currents in them are "balanced," *i.e.*, when $V_b = V_a$ and $I_b = I_a$, and the two voltages and two currents differ respectively in phase by 90 degrees, then $\theta_b = \theta_a$, and the total output is equal to twice the output of each phase, viz.,

$$P = 2\, V_a I_a \cos \theta_a \qquad (3a)$$

To transmit electric power from, or to, a four-wire two-phase machine, a four-wire transmission line is necessary. Such a system is equivalent to two independent single-phase systems supplied from a single generator. The line currents are equal to the "phase currents" (*i.e.*, to the currents in the respective windings), and the line voltages are equal to the "phase voltages" (*i.e.*, to the voltages between the terminals of the respective windings).

To transmit electric power from, or to, a three-wire two-phase machine, only three wires are necessary. The wire which is connected to the common terminal of the two phases serves as the return path for the currents supplied by both. This common wire is usually referred to as the "middle" wire, and the other two wires as the "outside" wires. The currents in the outside wires are equal to the phase currents, and the current in the middle wire is equal to the *resultant* of the phase currents. The voltage between the middle wire and each outside wire is equal to the corresponding phase voltage, and the voltage between the two outside wires is equal to the resultant of the phase voltages.

When the currents in the two phases are balanced sine-wave currents, the current in the middle wire is $\sqrt{2}$ times the current in each outside wire. This follows from the fact that the current in the middle wire is equal to the resultant of two vectors each equal in length to the phase current, and making with each other an

angle of 90 degrees. Similarly, when the phase voltages are balanced sine-wave voltages, the voltage between the two outside wires is $\sqrt{2}$ times each phase voltage. This follows from the fact that the voltage between the two outside wires is equal to the resultant of two vectors equal in length to the phase voltage and making an angle of 90 degrees with each other. Hence, for a balanced two-phase system:

$$\text{Current in middle wire} = \sqrt{2}I \qquad (4)$$
$$\text{Voltage between outer wires} = \sqrt{2}V \qquad (4a)$$

where I and V are respectively the r.m.s. values of the phase currents and phase voltages.

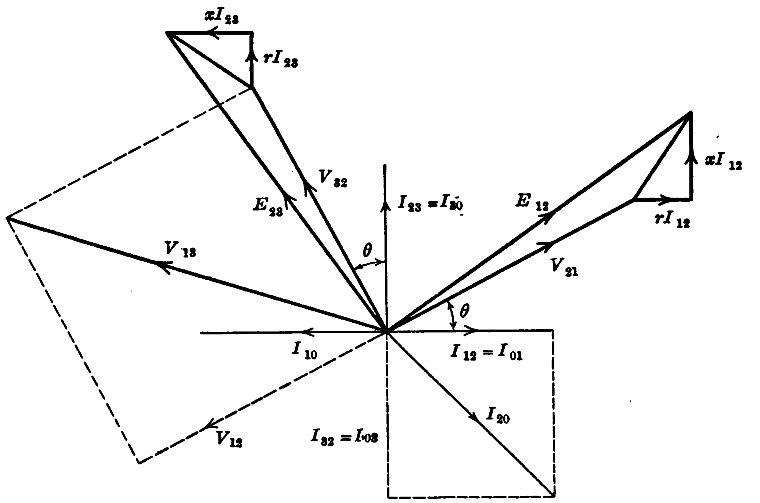


Fig. 146.—Vector diagram of three-wire two-phase alternator with balanced load.

A complete vector diagram for a three-wire two-phase generator is given in Fig. 146. The internal resistance and reactance of each phase are represented by the symbols r and x. The significance of the other symbols is indicated in Fig. 145*B*.

Problem 1.—(*a*) Referring to Fig. 145*A*, prove that the no-load terminal voltage between "adjacent" terminals of a four-wire two-phase generator, when the middle points of the two phases are connected to each other, is equal to $\frac{1}{\sqrt{2}}$ times the voltage between "diametrically" opposite terminals (*i.e.*, between the terminals of each phase). (*b*) Draw a vector diagram showing the phase voltages and the voltage between each pair of terminals. (Note: When the middle, or neutral, points of the two phases are not

connected, the voltage between adjacent terminals depends upon the insulation resistance and electrostatic capacity between the two phases and the circuits connected thereto.)

189. Unbalancing Effect of Middle Wire in a Three-wire Two-phase Transmission Line.—From the relations just deduced it may readily be shown that when a two-phase generator is connected by a three-wire transmission line to a two-phase receiver

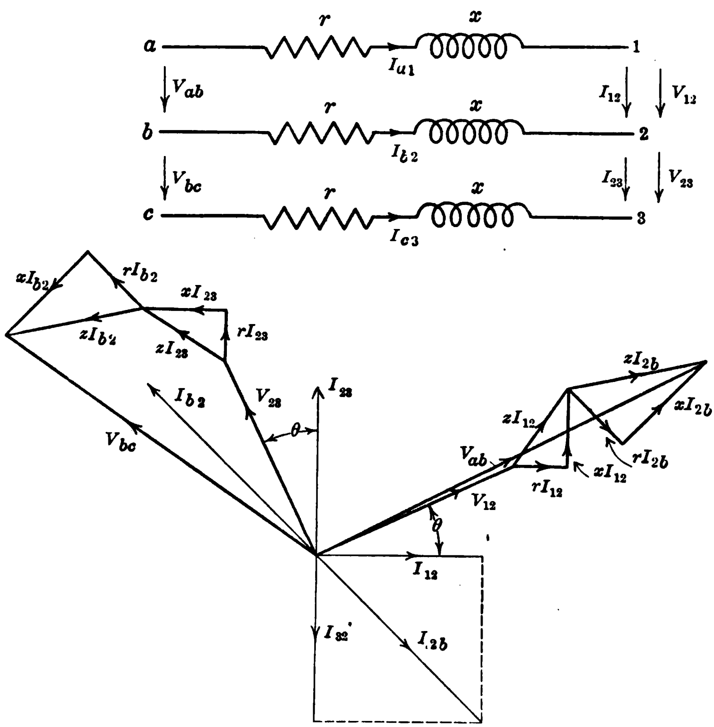


FIG. 147.—Vector diagram of three-wire two-phase transmission line.

(motor or lamps), the load supplied to the receiver and the load on the generator cannot *both* be balanced. This is due to the unsymmetrical effect, on the two phases, of the impedance drop in the middle wire.

A three-wire line is shown diagrammatically in the upper part of Fig. 147. Let r and x be the resistance and reactance respectively of each of the three-line wires, and let $z = \sqrt{r^2 + x^2}$ be the impedance of each of these wires. Imagine a two-phase generator connected to the left-hand end of the line, with its A phase between

a and b, and its B phase between b and c. Similarly imagine a two-phase *balanced* load connected to the other end of the line, with its A phase between 1 and 2, and its B phase between 2 and 3. Let θ be the power-factor angle of each phase of the load, *i.e.*, the angle by which each phase current lags the voltage drop through this phase in the direction of this current.

The terminal voltage V_{ab} of phase A of the generator will then be the vector sum of (1) the impedance drop zI_{12} in the wire $a1$, (2) the voltage drop V_{12} through phase A of the load, and (3) the impedance drop zI_{2b} in the wire $b2$ (or *less* vectorially the impedance drop zI_{b2}, since $I_{b2} = -I_{2b}$). Similarly, the terminal voltage V_{bc} of phase B of the generator is the vector sum of (1) the impedance drop zI_{b2} in the wire $b2$, (2) the voltage drop V_{23} through phase B of the load, and (3) the impedance drop zI_{23} in the wire $c3$ (or *less* vectorially the impedance drop zI_{c3}, since $I_{c3} = - I_{23}$).

The components of V_{ab} in the vector diagram are therefore V_{12}, rI_{12} parallel to I_{12}, xI_{12} leading I_{12} by 90 degrees, rI_{2b} parallel to I_{2b}, and xI_{2b} leading I_{2b} by 90 degrees, where I_{2b} is the vector sum of I_{12} and I_{32}. Similarly the components of V_{bc} in the vector diagram are V_{23}, rI_{23} parallel to I_{23}, xI_{23} leading I_{23} by 90 degrees, rI_{b2} parallel to I_{b2}, and xI_{b2} leading I_{b2} by 90 degrees, where I_{b2} is the vector sum of I_{21} and I_{23}. These relations are all shown in the vector diagram.

From the vector diagram it is evident that when the impedance drop in the middle wire is appreciable in comparison with the voltage across each phase of the load, the terminal voltages of the two phases of the generator will be unequal in r.m.s. values, and will differ in phase by an angle greater than 90 degrees. This, of course, is on the assumption above made that the loads on the two phases of the receiver are balanced. Conversely, when the electromotive forces generated in the two phases of the generator are equal and differ in phase by 90 degrees (which is the usual case), the voltages across the two phases of the receiver will be unequal in r.m.s. value and differ in phase by less than 90 degrees, and the currents in the two phases of the generator and receiver will be unequal in value and will not be exactly 90 degrees apart in phase.

Unequal voltages on the several phases of a polyphase system are always unsatisfactory from an operating point of view,

particularly when a part of the load (or all of it) consists of incandescent lamps. Such lamps are designed for operation at a definite voltage; a slight decrease in voltage lowers their candle-power by a relatively large amount, and a slight increase in voltage produces a relatively large decrease in their life (see the article on *Lamps, Incandescent Electric*, in Pender's *Handbook for Electrical Engineers*). Again, the output of a polyphase machine (for a given temperature rise) is less when the loads on its several phases are unbalanced than when these loads are balanced. For these reasons, the three-wire two-phase system is not as satisfactory as the three-phase system, which latter also requires but three transmission wires (see the next article).

Problem 2.—20,000 kilowatts is to be supplied from a power house to a substation 50 miles away, over a three-wire transmission line. The frequency is 25 cycles per second. The line wires are each No. 0000 A. W. G., spaced 6 feet between centers, and so arranged that they form the three edges of an equilateral prism. The resistance of each wire is 0.27 ohm per mile and, for the arrangement specified, the reactance of each wire is 0.30 ohm per mile. The power factor of the load on the substation (*i.e.*, of each phase of this load) is 90 per cent., with the current lagging. (Sine-wave currents and voltages are to be assumed.)

(*a*) Were the system a two-phase system what would be the r.m.s. value of each phase current, the phase voltage at the substation being 60,000? (*b*) What would be the current in the middle wire of the transmission line. (*c*) What would be the voltages between the "middle" bus-bar and each "outside" busbar in the power house in order to maintain 60,000 volts between the corresponding bus-bars at the substation? (*d*) What would be the power factor of each phase of the total load at the power house? (*e*) What would be the load on each phase at the power house? (*f*) What would be the voltage between the two outside bus-bars at the substation? (*g*) At the power house? (*h*) What would be the total power lost in the transmission line, both in kilowatts and in per cent. of the power delivered to the substation? (*i*) Draw to scale a complete vector diagram. [NOTE.—The voltages may be found graphically from the vector diagram, or analytically by resolving the several voltages into their components along two mutually perpendicular axes, and proceeding as explained in Article 2. The directions of the vectors representing the currents in the two phases of the load will be found the most convenient axes of reference; see Fig. 147.]

Answer.—(*a*) 185 amperes. (*b*) 262 amperes. (*c*) 68,400 volts and 65,900 volts respectively. (*d*) 90.4 per cent. and 85.5 per cent. respectively. (*e*) 11,420 kilowatts and 10,430 kilowatts respectively. (*f*) 84,900 volts. (*g*) 99,700 volts. (*h*) 1850 kilowatts or 9.25 per cent.

190. Three-phase Systems. Y and Δ Connections.—As noted in Article 185, a three-phase machine is one whose armature has

three independent windings, so arranged that the electromotive forces generated in them have the same r.m.s. value but differ in phase by 120 degrees. In general, the three windings, or

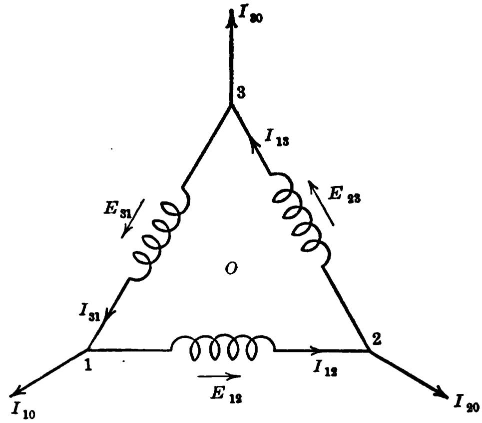


FIG. 148.—Delta connection.

phases, of a three-phase machine are permanently connected inside the machine in one of the two ways indicated in Figs. 148 and 149.

The connection shown in Fig. 148 is called a "mesh," or

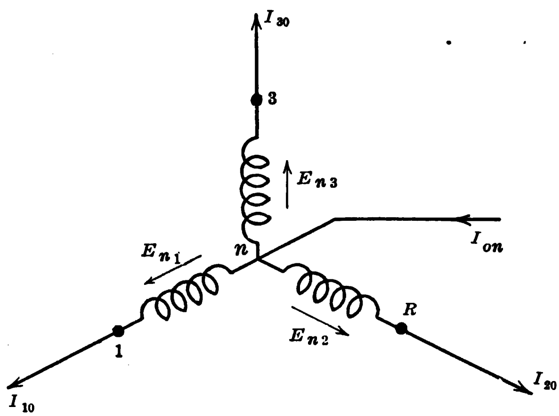


FIG. 149.—Y connection.

"delta," connection; the latter name arising from the similarity between the diagram and the Greek letter "Δ." The connection shown in Fig. 149 is called a "star," or "wye" connection; the

latter name arising from the similarity between the diagram (when inverted) and the letter "*Y*." In like manner, any three circuits connected as shown in Fig. 148 are said to be connected in Δ, and any three circuits connected as shown in Fig. 149 are said to be connected in *Y*.

With either type of connection but three terminals are needed to connect the machine (or group of circuits) to the transmission line over which energy is supplied from or to it. When the *Y* connection is employed the common junction point *n* of the three windings may also be connected to a terminal, and a fourth wire may be used in the transmission line. This junction point is called the "neutral point" of the connection, and the fourth wire the "neutral wire."

In a Δ connection (Fig. 148) it is evident that the current in each line wire is not equal to the current in each winding, either in r.m.s. value or phase, but is the *resultant* of the currents in the two windings connected to this terminal. That is, **in a Δ connection the line currents are NOT equal to the phase currents.** On the other hand, the voltages between the line wires of a Δ connection are the same as the voltages across the corresponding phases, or windings.

In a *Y* connection (Fig.149) the relations are just the opposite. The line currents (in the three main wires) and phase currents are equal, but the line voltages between the main wires are the *resultant* of the voltage drops through the two windings connected to the two lines under consideration. That is, **in a Y connection the line voltages are NOT equal to the phase voltages.** From Fig. 149 it is evident that the current in the neutral wire (when such a wire is provided) is the *resultant* of the currents in the three phases, and the voltage between each line wire and the neutral is equal to the phase voltage of the particular phase between this wire and the neutral.

191. Power Output and Input of Three-phase Apparatus.—The electric power output (or input) of a three-phase machine is always equal to the total power outputs (or inputs) of its three phases, irrespective of the type of connection employed and irrespective of whether or not the currents and voltages are balanced. That is, calling V_a, V_b and V_c the r.m.s. values of the PHASE voltages, I_a, I_b, and I_c, the r.m.s. values of the PHASE currents, and θ_a, θ_b and θ_c the angles by which the respective

PHASE currents lag the PHASE voltages, then the total power output (or input) is

$$P = V_a I_a \cos \theta_a + V_b I_b \cos \theta_b + V_c I_c \cos \theta_c \qquad (5)$$

When the currents and voltages are *balanced, i.e.*, when $V_a = V_b = V_c$ and $I_a = I_b = I_c$, and the three voltages are 120 degrees apart in phase, and the three currents are likewise 120 degrees apart in phase, then $\theta_a = \theta_b = \theta_c$, and the total power output (or input) is

$$P = 3V_p I_p \cos \theta_p \qquad (5a)$$

where V_p is the r.m.s. value of each PHASE voltage, I_p the r.m.s. value of each PHASE current, and θ_p is the angle by which each PHASE current lags the corresponding PHASE voltage.

As will be shown later (Articles 194 and 195), for a balanced load and sine-wave currents and voltages, the phase voltage V_p for a Y connection is equal to $\frac{1}{\sqrt{3}}$ times the line voltage V_l, and the phase current I_p for a Δ connection is equal to $\frac{1}{\sqrt{3}}$ times the line current I_l. In a Δ connection the line voltages and the phase voltages are always equal, and in a Y connection the line currents and the phase currents are always equal. Hence, for either kind of connection, the above expression for the power output (or input) for a BALANCED load may be written

$$P = \sqrt{3}\ V_l I_l \cos \theta_p \qquad (5b)$$

This expression is usually written without subscripts, viz.,

$$P = \sqrt{3}\ VI \cos \theta \qquad (5c)$$

but the significance of each symbol is always as just stated, viz.:

(1) V is the line voltage, *i.e.*, the voltage read by a voltmeter connected between any two of the three main terminals of the machine (not between a main terminal and the neutral).

(2) I is the line current, *i.e.*, the current read by an ammeter connected in series with any one of the main line wires (not in the neutral).

(3) θ is the angle by which the current in any one of the three windings or phases lags the net voltage drop (or voltage rise) *in this particular winding*. It is particularly important to keep

in mind that the angle θ is NOT the difference in phase between the LINE current I and LINE voltage V.

When the voltage of a three-phase generator or receiver is specified, this voltage is always the voltage between terminals, or line voltage, irrespective of whether the machine is Δ or Y connected. By the power output (or input) of a three-phase machine is always meant the total power of all three phases. Similarly, by the kilovolt-ampere output or input is meant the total kilovolt-amperes of all three phases. In terms of the rated power P (in watts) and the rated voltage V, the rated current of a three-phase generator or receiver for a balanced load, is

$$I = \frac{P}{\sqrt{3}\ V \cos \theta} \tag{5d}$$

where $\cos \theta$ is the power factor (expressed as a fraction). The corresponding volt-ampere output or (input) is then

$$\text{Volt-amperes} = \sqrt{3}VI \tag{5e}$$

When the generator or receiver is Y connected, the current given by equation (5*d*) is the current in each phase of the load. When the load is Δ connected, the current in each phase of the load is equal to $\frac{1}{\sqrt{3}}$ times this current (provided the currents are sinusoidal).

Problem 3.—The load of 20,000 kilowatts specified in Problem 2, Article 189, is transmitted by three-phase currents (instead of two-phase, as there considered). All other conditions at the substation are the same, *i.e.*, the voltage between bus-bars at the substation is 60,000, the load is balanced, and the power factor is 90 per cent., with the current lagging. The transmission line is the same as in Problem 2. (Sine-wave currents and voltages are to be assumed.)

(*a*) What is the kilowatt load supplied to each phase of the substation apparatus? (*b*) What is the kilovolt-ampere load of each phase of the substation apparatus? (*c*) What is the current in each wire of the transmission line? Compare with (*a*) and (*b*) of Problem 2. (*d*) What is the total power loss in the three wires of the line, both in kilowatts and in per cent. of the power delivered? Compare with (*h*) of Problem 2. (*e*) Were this same load transmitted over a single-phase line of the same length and *same weight* of copper, with the same voltage between wires at the substation, what would be the power lost in the line, both in kilowatts and in per cent. of the delivered power? Compare with (*d*). (*f*) Were the transformers in the substation three single-phase transformers with their primary windings connected in Y, what would be the voltage across the terminals of each?

(*g*) What would be the current in the primary of each transformer? (*h*) What would be the kilowatt input to each transformer? (*i*) What would be the difference in phase between the primary terminal voltage and the primary current of each transformer? (*j*) Were the primary windings of the transformers connected in Δ, what would be the voltage across each? (*k*) What would be the current in the primary winding of each transformer? (*l*) What would be the kilowatt input to each? Compare with (*h*). (*m*) What would be the difference in phase between the terminal voltage and primary current in each transformer? Compare with (*i*).

Answer.—(*a*) 6667 kilowatts. (*b*) 7407 kilovolt-amperes. (*c*) 214 amperes, as compared with 185 amperes in each outside wire and 262 amperes in the middle wire of the two-phase line. (*d*) 1850 kilowatts or 9.25 per cent., the same as for two-phase transmission over a three-wire line. [Note, however, that in the two-phase line the voltage between outside wires, at the substation, is 84,900 volts, whereas in a three-phase line *each wire* differs in potential from *each of the others* by the *same* amount, viz., 60,000 volts.] (*e*) 2470 kilowatts, or 12.35 per cent. This loss is 33.3 per cent. greater than the loss in a three-phase or two-phase three-wire line of the same total weight of copper. (*f*) 34,600 volts. (*g*) 214 amperes. (*h*) 6667 kilowatts. (*i*) 25.8 degrees. (*j*) 60,000 volts. (*k*) 123.5 amperes. (*l*) 6667 kilowatts, the same as when the transformers are *Y* connected. (*m*) 25.8 degrees, the same as when the transformers are *Y* connected.

192. Resultant Electromotive Force Acting Around Loop Formed by a Δ Connection.—When the electromotive forces in

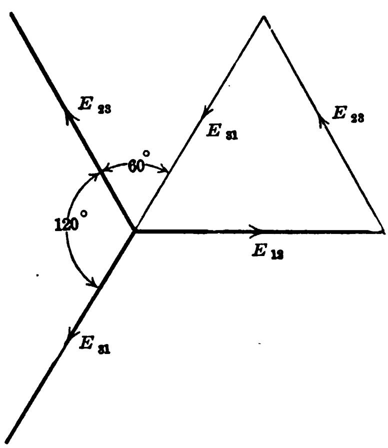


Fig. 150.

the three phases of a Δ connection are balanced sine-wave electromotive forces, they may be represented by three vectors of equal length and making angles of 120 degrees with each other, as shown in Fig. 150. Note that these three vectors represent

the electromotive forces which act in the *same direction around* the loop formed by the Δ; compare with Fig. 148. The resultant electromotive force tending to cause a current to flow *around* the Δ is then the vector sum of these three vectors. But this vector sum is zero, since the three vectors when laid off end to end, as shown in Fig. 150, form a closed polygon, *i.e.*, an equilateral triangle.

Hence, when no current enters or leaves the Δ at its junction points, *i.e.*, when no power is supplied by or to the Δ, there is no current in any of the windings. This, however, is not true when the wave shapes of the electromotive forces contain harmonics whose frequencies are 3 times, or any multiple of 3 times, the fundamental frequency (see Problem 3, Article 213).

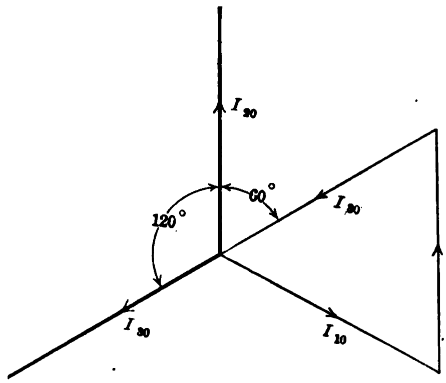


FIG. 151.

193. Current in Neutral Wire of a Y Connection.—When the currents in the three phases of a Y connection are balanced sine-wave currents, they may be represented by three vectors of equal length and making angles of 120 degrees with each other, as shown in Fig. 151. Note that these vectors represent the currents in the *same relative direction with respect to the neutral point* of the Y, *i.e.*, either all outward from the neutral point, or all inward toward the neutral point; compare with Fig. 149. The resultant current in the neutral wire, if any, is then the vector sum of these three vectors. This vector sum is zero, since the three vectors when laid off end to end, as shown in Fig. 151, form a closed polygon.

Hence, when the currents in a Y-connected generator, or a

Y-connected receiver, are balanced and their wave shapes are sine-waves, there is no current in the neutral wire connected to it. Consequently, when the load on a three-phase system is balanced, the neutral wire is without effect, provided the currents are sinusoidal. The neutral of a Y connection is, however, often brought out to a terminal, usually for the purpose of connecting it to ground, so that the voltage between each of the other terminals and the ground will have the same r.m.s. value. When the neutral of a Y is accessible it is also possible to obtain, for the operation of lamps or other single-phase apparatus, a voltage equal to the phase voltage, which is $\frac{1}{\sqrt{3}}$ times the line voltage (see Article 194).

When a Y-connected generator with neutral wire supplies an *unbalanced* load, the current in the neutral wire will not be zero. Instead, a current will flow in the neutral tending to equalize the currents in the three phases of the generator. Again, even though the currents in the three phase are balanced, a current of triple-frequency will flow in the neutral wire in case the currents have a wave shape containing a third harmonic (see Problem 4, Article 213).

194. Relation Between Line Voltage and Phase Voltage for a Y Connection.—When the terminal voltages of the three phases of a Y connection are balanced sine-wave voltages, they may be represented (Fig. 152) by three vectors V_{1n}, V_{2n}, and V_{3n}, equal in length and making angles of 120 degrees with each other. These vectors represent the drop of potential from the respective terminals to the neutral point of the Y.

The line voltage V_{12} between terminals 1 and 2 is then the resultant drop of potential from 1 to 2, which in turn is equal to the drop of potential V_{1n} from 1 to the neutral, plus vectorially the drop of potential V_{n2} *from* the neutral *to* 2. Since V_{n2} is equal in length to V_{2n}, but in the *opposite* direction, the voltage drop V_{12} from 1 to 2 is the line marked V_{12} in Fig. 152. Similarly, the vectors which represent the voltage drops from 2 to 3 and from 3 to 1 are the lines marked V_{23} and V_{31}.

Hence, calling V_p the r.m.s. value of the voltage between each terminal and the neutral, *i.e.*, the *phase* voltage of the Y, then, from Fig. 152, the r.m.s. value of each line voltage is

$$V_l = 2V_p \cos 30° = \sqrt{3}\, V_p \qquad (6)$$

From Fig. 152 it is also evident that the line voltages differ in phase by 120 degrees. Hence the line voltages due to balanced sine-wave phase voltages in a Y connection are also balanced sine-wave voltages.

Conversely, when balanced sine-wave line voltages of r.m.s. value V_l are impressed on a balanced Y connection, the voltages across each phase of the Y, *i.e.*, between each terminal of the Y and the neutral point are also balanced and have the r.m.s. value

$$V_p = \frac{1}{\sqrt{3}} V_l \tag{6a}$$

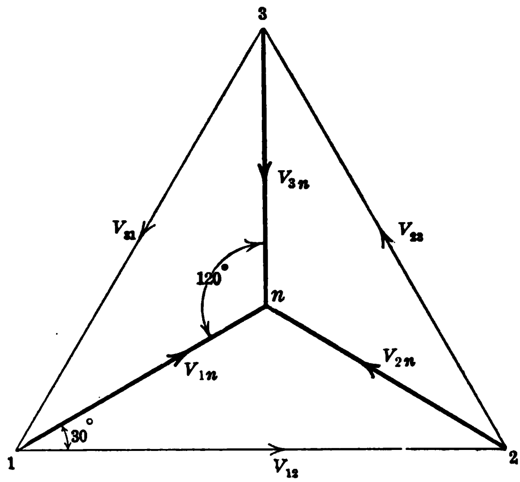


Fig. 152.—Line voltages and phase voltages for Y connection.

By a *balanced* Y is meant a Y connection of three windings all three of which have the same resistance and reactance, and in which the electromotive forces (if any) are equal in r.m.s. value and differ in phase by 120 degrees.

It should be carefully noted that the above relations hold only for *balanced sine-wave* voltages. When the voltages are not equal in r.m.s. value, the symmetrical relations represented by the equilateral triangle in Fig. 152 no longer hold. The analysis of unbalanced conditions is more complex, but may be readily effected by the use of the so-called "symbolic method" described in Chapter XVII. Again, in the particular case, which frequently occurs in practice, when the phase voltage contains a third harmonic (see Article 213), this third harmonic does

not appear in the line voltage, and consequently the r.m.s. value of the line voltage is less than $\sqrt{3}$ times the phase voltage.

Problem 4.—Two equal impedance coils are connected in series between two of the terminals of a Y-connected, 200-volt generator. The impedance of these coils is so high that the current supplied by the generator produces a negligible impedance drop in its windings.

(*a*) What is the r.m.s. value of the voltage between the junction of the two coils and the neutral point of the generator? (*b*) What is the r.m.s. value of the voltage between this junction point and the third terminal of the generator? (*c*) Draw to scale a complete vector diagram, showing the phase relations between these two voltages and the phase and line voltages.

Answer.—(*a*) 57.5. (*b*) 172.5 volts.

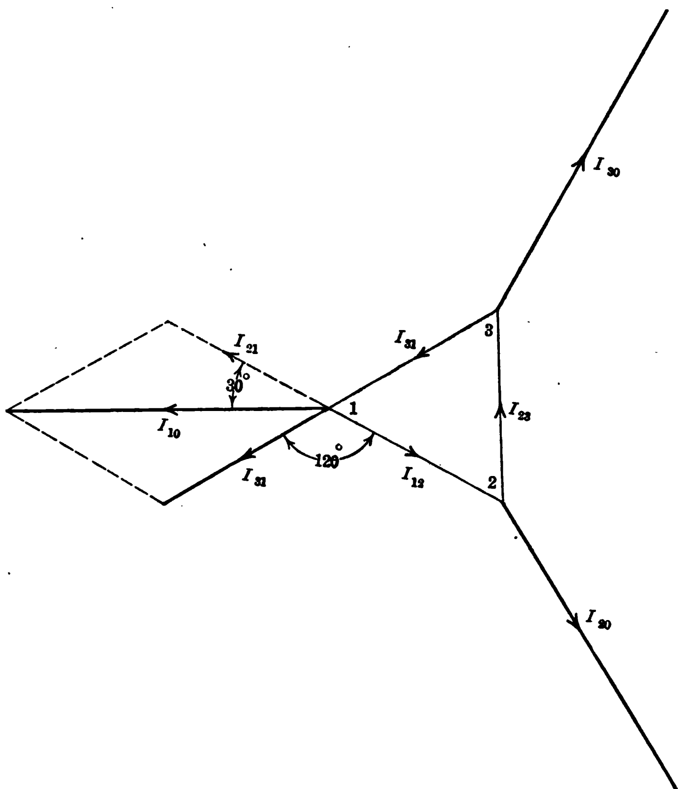


FIG. 153.—Line currents and phase currents for Δ connection.

195. Relation Between Line Current and Phase Current for a Δ Connection.—When the currents in the three phases of a Δ connection are balanced sine-wave currents, they may be represented (Fig. 153) by three vectors I_{12}, I_{23} and I_{31}, equal in

length and making angles of 120 degrees with each other. The line current I_{10} in line wire No. 1 *outward* from the Δ is equal to the resultant of the currents *toward* this terminal in the two windings connected to it.

From the figure it is evident that this resultant is the *vector difference* of I_{31} and I_{12}, since I_{31} is *toward* the terminal (junction point) and I_{12} is *away from* the terminal. Or, since $I_{12} = -I_{21}$, where I_{21} is the current *toward* the junction point in the branch 12 of the Δ, it follows that I_{10} is equal to the vector sum of I_{31} and I_{21}, as indicated in the figure. Hence, calling I_p the r.m.s. value of the *phase* current in the Δ, and I_l the r.m.s. value of the *line* current, then, from Fig. 153,

$$I_l = 2I_p \cos 30° = \sqrt{3}\ I_p \tag{7}$$

From Fig. 153 it is also evident that the line currents differ in phase by 120 degrees. Hence the line currents produced due to balanced sine-wave phase currents in a Δ connection are also balanced sine-wave currents.

Conversely, when the line currents are balanced sine-wave currents of r.m.s. value I_l, the phase currents in the Δ which supplies or absorbs the power transmitted are also balanced and have the r.m.s. value

$$I_p = \frac{1}{\sqrt{3}} I_l \tag{7a}$$

The above relations, like those between the phase voltage and line voltages of a Y hold only for *balanced sine-wave* currents. In particular, the phase current in a Δ connection often contains a third harmonic, yet this third harmonic current does not appear in the line current (see Article 213). Under such conditions the r.m.s. value of line current is less than $\sqrt{3}$ times the phase current.

Problem 5.—A 500-volt, 25-cycle, 100-kilovolt-ampere, three-phase Δ-connected generator supplies full load to a balanced three-phase circuit whose power factor is 85 per cent.

(*a*) What is the current in each of the three windings of the generator? (*b*) Were *one* of these windings opened, so that no current could flow through it, would the generator still supply power to the load? (*c*) Neglecting the internal impedance of the generator windings, would the line currents remain balanced? (*d*) Would the line voltages remain balanced? (*e*) What would be the maximum balanced load at 85 per cent. power factor that the generator could supply under the conditions specified in (*b*) without exceeding its kilovolt-ampere rating? (*f*) What would be the power factor

of the load supplied by each of these windings? (*g*) What proportion of this load would be supplied by each of the two windings in service? (*h*) Draw to scale a complete vector diagram for (*e*), showing the line currents, the line voltages, and the currents in the two windings of the generator.

Answer.—(*a*) 66.7 amperes. (*b*) Yes. (*c*) Yes. (*d*) Yes. (*e*) 49.1 kilowatts. (*f*) 47.3 per cent. for one phase and 100 per cent. for the other phase. (*g*) 32.1 per cent. for one phase and 67.9 per cent. for the other.

196. Balanced Three-phase Circuit Equivalent to Three Single-phase Circuits.—When both the generator and the load are Y connected (see Fig. 154), a wire connecting the neutral points of the two will have no current in it *provided* the line currents (which in this case are equal to the phase currents) are *balanced and have sine-wave shapes.* From symmetry, it is evident that this condition will obtain when

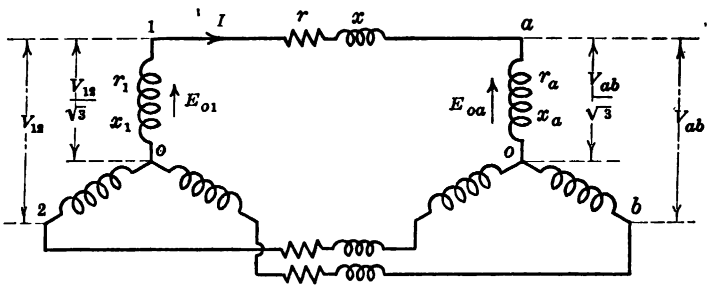


FIG. 154.

(1) The electromotive forces generated in the three phases of the generator are balanced,

(2) The electromotive forces generated in the three phases of the receiver are balanced,

(3) Each phase of the generator has the same resistance and the same reactance,

(4) Each phase of the receiver has the same resistance and the same reactance,

(5) Each wire of the line has the same resistance and the same reactance.

In practice these conditions are usually satisfied, at least to a close approximation. Moreover, as will be shown in the next Article, a Δ-connected generator or receiver may always be considered as replaced by an "equivalent" Y-connected generator or receiver. Consequently, any problem concerning the current, voltage and power relations in *balanced* three-phase

circuits may always be reduced to the solution of a network of the type shown in Fig. 154, provided the currents and voltages are sinusoidal.

As will now be shown, a three-phase circuit of the type shown in Fig. 154 may always be considered equivalent to three independent single-phase circuits, provided the currents and voltages are sinusoidal and balanced. Hence, *every problem in regard to balanced three-phase circuits carrying sine-wave currents and voltages may be reduced to the solution of a single-phase circuit.*

This proposition follows from the simple fact that since no current can exist in the neutral of the balanced circuit shown in Fig. 154 (always assuming the currents and voltages to be sinusoidal), no difference of potential can exist between the neutral points of the two Y's. Hence, each phase of this circuit may be treated as a single-phase circuit in which the "return wire" of the transmission line has *zero resistance and zero reactance.* That is, each phase of the three-phase circuit shown in Fig. 154 is equivalent to the single-phase circuit shown in Fig. 155.

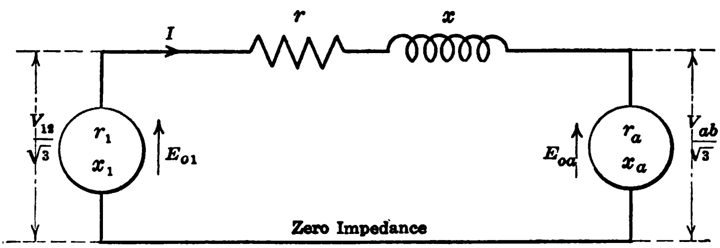


FIG. 155.

In the reduction of a three-phase problem to a single-phase one, the following relations must be carefully noted:

(1) In the equivalent single-phase circuit the generated electromotive force of the generator, and the back electromotive force (if any) in the receiver are respectively equal to the electromotive force *per phase* of the corresponding Y.

(2) The internal resistance and reactance of the generator and receiver in the single-phase circuit are equal respectively to the resistance and reactance *per phase* of the corresponding Y.

(3) The resistance and reactance of the transmission line in the single-phase circuit are equal respectively to the resistance and reactance of *one wire* of the three-phase transmission line.

(4) The terminal voltages of the generator and receiver in the single-phase circuit are equal respectively to the *voltages to neutral* in the corresponding Y (*i.e.*, the *phase* voltages of the Y). The corresponding voltages *between wires* in the three-phase circuit are therefore equal to $\sqrt{3}$ times the terminal voltages for the single-phase circuit.

(5) The current in the single-phase circuit is equal to the *line* current in the three-phase circuit, or to the *phase* current in the Y's.

(6) The angle by which the current in the single-phase circuit lags (or leads) any voltage is the same as the angle by which the *phase* current in the three-phase circuit lags (or leads) the corresponding *phase* voltage. This angle is NOT the angle between the *line* voltage and the *line* current in the three-phase circuit.

(7) The power output of the generator, the power input to the receiver, and the power lost in the line in the single-phase circuit are equal respectively to $\frac{1}{3}$ the total output of the three-phase generator, $\frac{1}{3}$ the total input to the three-phase receiver, and $\frac{1}{3}$ the total power lost in the three wires of the transmission line.

Problem 6.—Referring to Problem 3, Article 191 (*a*) what is the voltage between the power-house bus-bars, and (*b*) what is the power-factor of the total load on the power-house? Compare with (*c*) and (*d*) of Problem 2, Article 189. (*c*) Draw to scale a complete vector diagram showing (1) the three line currents, (2) the three voltages to neutral at both ends of the transmission line, and (3) the three line voltages at both ends of the transmission line.

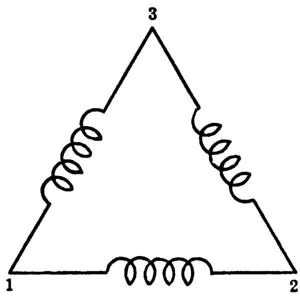
As stated in Problem 3, the load on the substation is 20,000 kilowatts, the voltage between the substation bus-bars is 60,000 volts, and the power factor of this load is 90 per cent., the current lagging. The transmission line is 50 miles long, the resistance of each wire is 0.27 ohm per mile, and the reactance of each wire is 0.303 ohm per mile (at 25 cycles per second).

Answer.—(*a*) 67,000 volts between *each* pair of wires, as against 68,400 volts between one outside wire and the middle wire, 65,900 volts between the other outside wire and the middle wire, and 99,700 volts between the two outside wires for the two-phase system. (*b*) 88.1 per cent. for *each* phase, as against 90.4 per cent. for one phase and 85.5 per cent. for the other phase for the two-phase system.

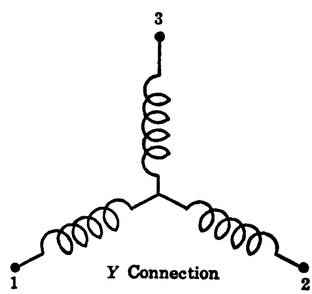
197. Balanced Δ Equivalent to a Balanced Y.—From the relations deduced in Articles 194 and 195, it follows that the conditions under which a Δ-connected generator or receiver may be considered as replaced by an equivalent Y are the following (see Fig. 156):

(1) The currents and voltage must be sinusoidal and balanced.

(2) The electromotive force developed in each phase of an equivalent Y must be taken equal to $\frac{1}{\sqrt{3}}$ times the electromotive force per phase of the Δ which it replaces. This follows from the fact that each phase of a Δ is connected across two of the



Line Voltage	=	V
Line Current	=	I
Phase Voltage	=	V
Phase Current	=	$\frac{I}{\sqrt{3}}$
E. M. F. per Phase	=	E
Resistance per Phase	=	r
Reactance per Phase	=	x



Line Voltage	=	V
Line Current	=	I
Phase Voltage	=	$\frac{V}{\sqrt{3}}$
Phase Current	=	I
E. M. F. per Phase	=	$\frac{E}{\sqrt{3}}$
Resistance per Phase	=	$\frac{r}{3}$
Reactance per Phase	=	$\frac{x}{3}$

FIG. 156.—Δ connection and its equivalent Y.

line wires, and consequently, when there is no current flowing, its electromotive force is equal to the line voltage. On the other hand, the line voltage for a Y when there is no current flowing is equal to $\sqrt{3}$ times the electromotive force developed in each of its phases.

(3) The resistance of each phase of the equivalent Y must be taken equal to $\frac{1}{3}$ the resistance of the Δ which it replaces. This follows from the fact that the phase current in a Y is equal to the line current, whereas the phase current in a Δ is equal to $\frac{1}{\sqrt{3}}$ times the line current. Hence, since the power dissipated in

a resistance is proportional to the *square* of the current, the resistance of each phase of the Y must, for the same power loss, be $\left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$ the resistance of the Δ which it replaces.

(4) The reactance of each phase of the equivalent Y must likewise be taken equal to $\frac{1}{3}$ the reactance of the Δ which it replaces. This follows from the fact that, for the same difference in phase between the phase voltages and currents in the equivalent Y as in the Δ which it replaces, the *ratio* of the reactance to the resistance must be the same. Hence, since the resistance of the equivalent Y is $\frac{1}{3}$ the resistance of the Δ which it replaces, the reactance of the equivalent Y must likewise be equal to $\frac{1}{3}$ the reactance of the replaced Δ.

(5) When all Δ connections have been replaced by equivalent Y connections, the phase current for any Δ which has been replaced by an equivalent Y will be equal to $\frac{1}{\sqrt{3}}$ times the phase current of this Y, and the phase voltage for the Δ will be equal to $\sqrt{3}$ times the phase voltage for corresponding Y.

Since a balanced Δ may be replaced by an equivalent balanced Y, a balanced Δ may be considered as having a "neutral point," meaning thereby the point whose potential is the same as the potential of the neutral point of the equivalent Y. This point cannot, of course, be located on any of the conductors which form the actual Δ, but is somewhere in the space surrounding the windings.

198. Two-Phase Three-phase Transformation.—As shown in Problem 6, Article 196, three-phase transmission is more satisfactory than two-phase three-wire transmission. Consequently, whenever two-phase apparatus is employed, it is the usual practice to transmit the electric energy from, or to, this apparatus as three-phase energy, converting it from two-phase to three-phase at the generator, or from three-phase to two-phase at the receiver. Such a conversion is readily effected by means of two transformers connected as shown in Fig. 157.

The two transformers have the same number of turns in the windings which are connected to the two-phase generator, or receiver. The second winding of one of the transformers, the one marked A in the figure, has a tap brought out from its

middle point. The second winding of the other transformer B, has a number of turns equal to $\sqrt{3/4} = 0.866$ times the number of turns in the second winding of the first transformer. These latter windings are connected to each other and to the three-phase transmission line as shown in the figure.

When the transformers are used to convert from two-phase to three-phase, and balanced two-phase voltages are impressed across the terminals a and b and b and c, two-phase electromotive forces are induced in the secondary windings (by mutual inductance). However, the induced electromotive force in the secondary of transformer B will have an r.m.s. value equal to $\sqrt{3/4}$

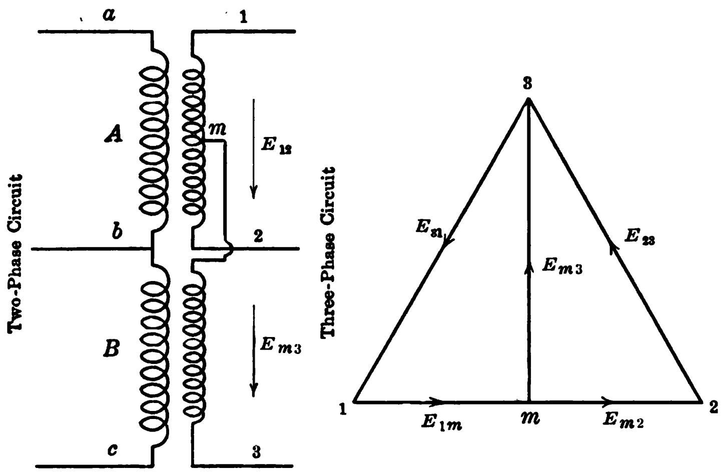


FIG. 157.—Two-phase three-phase transformation.

times the r.m.s. value of the electromotive force induced in the secondary winding of A. The vector diagram for these two electromotive forces is then as shown in the figure.

Let E be the r.m.s. value of the electromotive force induced in the secondary of transformer A. Then the r.m.s. value of E_{m3}, viz., the electromotive force induced in the secondary of B is $\sqrt{3/4}E$. The electromotive forces E_{1m} and E_{m2} induced in the two halves of the secondary winding of A, each have an r.m.s. value equal to $1/2E$. The electromotive force E_{12} between terminals 1 and 2 then has an r.m.s. value E. The resultant electromotive force E_{23} between 2 and 3 is the vector sum of E_{2m} and E_{m3}, which has an r.m.s. value equal to $\sqrt{(1/2E)^2 + (\sqrt{3/4}E)^2}$

$=E$, and leads E_{12} by 120 degrees, as shown in the figure. In like manner, the resultant electromotive force E_{31} between 3 and 1 is the vector sum of E_{3m} and E_{m1}, which has an r.m.s. value equal to $\sqrt{(\frac{1}{2}E)^2 + (\sqrt{\frac{3}{4}}E)^2} = E$, and leads E_{23} by 120 degrees, as shown.

Hence, for no load on the transmission line, the voltages between 1 and 2, 2 and 3, and 3 and 1 are *balanced* three-phase voltages. When the transformers are properly designed, the terminal voltages on both the primary and secondary side will also remain balanced under load, provided the load currents are balanced.

In exactly the same manner it may be shown that when the transformers are used to convert from three-phase to two-phase, balanced two-phase voltages will be produced at the secondary terminals when balanced three-phase voltages are impressed on the three-phase terminals.

The arrangement here described for converting from two-phase to three-phase, or conversely, is known as the "*T*," or Scott connection.

A particularly interesting example of its use is in the oldest of the Niagara power-houses (that of the Niagara Falls Power Co.) the generators in which are two-phase.

Problem 7.—(*a*) Referring to Fig. 157, prove that when the three-phase load supplied by the two transformers is balanced, each transformer supplies one-half of the load. (*b*) Prove that, under the same conditions, the kilovolt-ampere output of transformer B is only 87 per cent. of the kilovolt-ampere output of A. (*c*) Show that if N is the total number of turns in the secondary of A, and I is the r.m.s. value of the three-phase line current, the r.m.s. value of the resultant secondary ampere-turns of A is $0.87NI$, that is, the same as the r.m.s. value of the secondary ampere-turns of B.

199. Measurement of Power in a Three-wire Circuit.—In a two-phase three-wire circuit, the current in each of the outside wires a and b in Fig. 158 is equal to the current in the particular phase of the load to which it is connected, and the voltage between each outside wire and the middle wire c is the voltage across this phase. Hence, each of the two wattmeters connected as shown in the figure will indicate the load on the particular phase to which it is connected, and the sum of the two wattmeter readings will give the total power input to the load.

As will be shown below, the *algebraic* sum of the readings

of two wattmeters connected as shown in Fig. 158 is *always* equal to the total power input to the load supplied by the three wires (provided there are *only* three wires, *e.g.*, no neutral wire). This statement is true whether the load is two-phase (Fig. 158) or three-phase (Fig. 160), balanced or unbalanced, and holds irre-

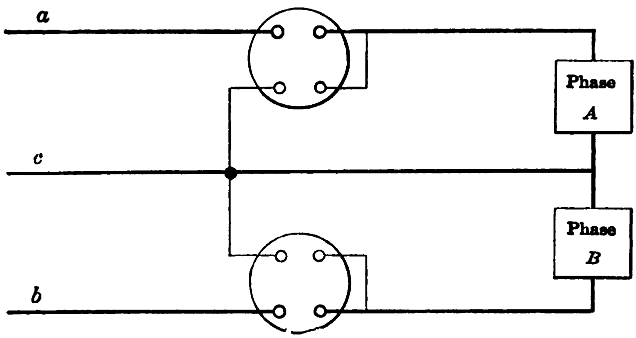


FIG. 158.—Two wattmeters and two-phase load.

spective of the wave shapes of the currents and voltages. Of course, when the load is three-phase (Fig. 160), the separate readings of the wattmeters do not give the load on any particular phase.

In the particular case of a perfectly *balanced* three-phase load, the total power may also be determined by measuring the power

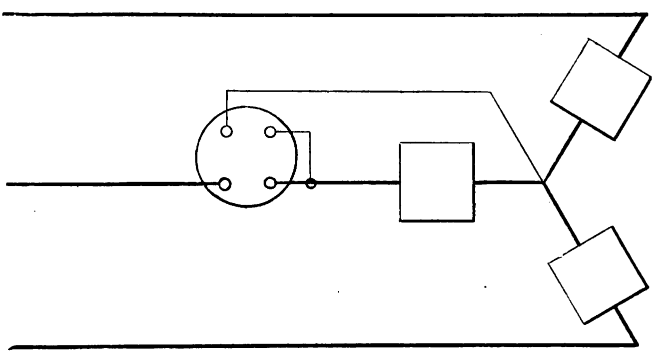


FIG. 159.—One wattmeter and three-phase load.

input to one phase and multiplying this by 3. To measure the power input to one phase it is of course necessary that the current circuit and voltage circuit of the wattmeter be connected in such a manner that (1) the current through its current coil be the

current in this particular phase of the load and (2) the voltage across its voltage coil be the voltage across this particular phase. This may be accomplished by connecting the current circuit of the wattmeter in series with one of the mains supplying the load, and the potential circuit of the wattmeter between one terminal of the load and the *neutral,* as shown in Fig. 159. When there is no neutral point accessible, one may be readily established by the use of a so-called "Y-box" (see the article on *Wattmeters* in Pender's *Handbook for Electrical Engineers*).

It is particularly important to note that when this single-wattmeter method is employed, the potential circuit of the wattmeter must be connected between one of the main terminals of the load and the neutral; *NOT between two main terminals.* That this is true follows from the fact that the reading of a wattmeter gives the *average* value of the product of (1) the instantaneous voltage impressed on the terminals of its potential circuit and (2) the instantaneous value of the current in its current circuit. This average value, as shown in Article 167, is equal to the product of (1) the r.m.s. value of the impressed voltage, (2) the r.m.s. value of the current, and (3) the cosine of the angle by which this current and voltage differ in phase. As noted in Article 191, when the load is Δ connected, the line current is not the phase current, and when the load is Y connected, the line voltage is not the phase voltage. Moreover, in neither case is the difference in phase between the line voltage and the line current equal to the difference in phase between the phase voltage and the phase current.

A three-phase load is seldom sufficiently well balanced to render the single-wattmeter method an accurate one. In most practical measurements the two-wattmeter method is employed. The two wattmeters are connected as shown in Fig. 160. The current circuit of wattmeter A is connected in series with one line wire a, and the current circuit of wattmeter B is connected in series with the second line wire b. The potential circuit of A is connected across from the wire a to the third line wire c, and the potential circuit of B is connected across from the wire b to the wire c.

If it be assumed, as is substantially true in a well-designed wattmeter, that its indications give the *average* value of the product of the instantaneous current through its current circuit and the

instantaneous voltage impressed on its potential circuit, it may be readily shown that the *algebraic* sum of the readings of the two wattmeters, when connected as shown in Fig. 160, is equal to the average power input to all three phases of the load, irrespective of whether the load is balanced or unbalanced, and irrespective of the wave shapes of the currents and voltages. Referring to Fig. 160, let the *instantaneous* values of the currents and potential drops be represented by *small* letters, with subscripts indicating their directions.

Wattmeter A then reads the *average* of $(v_{13}i_{a1})$, and wattmeter B reads the *average* of $(v_{23}i_{b2})$. Either of these averages may

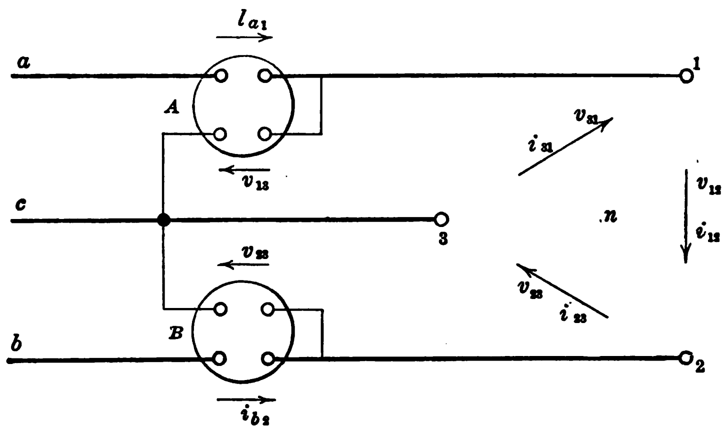


FIG. 160.—Two wattmeters and three-phase load.

be positive or negative quantities. Hence the combined readings of the two meters must be considered as an *algebraic* sum.

Consider first the case when the load is Δ connected. Then, from Fig. 160,

$$
\begin{aligned}
v_{13} &= -v_{31} \\
i_{a1} &= i_{12} - i_{31} \\
i_{b2} &= i_{23} - i_{12} \\
v_{12} &= -(v_{23} + v_{31})
\end{aligned}
$$

Hence, the algebraic sum of the readings of the two meters is the average of

$$
\begin{aligned}
v_{13}i_{a1} + v_{23}i_{b2} &= -v_{31}(i_{12} - i_{31}) + v_{23}(i_{23} - i_{12}) \\
&= -(v_{23} + v_{31})\, i_{12} + v_{23}i_{23} + v_{31}i_{31} \\
&= v_{12}i_{12} + v_{23}i_{23} + v_{31}i_{31} \qquad (8)
\end{aligned}
$$

But the average of $(v_{12}i_{12} + v_{23}i_{23} + v_{31}i_{31})$ is the total power input to the three phases of the load. Since in this deduction no assumption is made in regard to the wave shapes or whether the load is balanced or not, the algebraic sum of the wattmeter readings gives the total power input irrespective of the wave shapes and of the nature of the load.

Similarly, when the load is Y connected,

$$\begin{aligned} v_{13} &= v_{1n} + v_{n3} = v_{1n} - v_{3n} \\ v_{23} &= v_{2n} + v_{n3} = v_{2n} - v_{3n} \\ i_{a1} &= i_{1n} \\ i_{b2} &= i_{2n} \\ i_{3n} &= -(i_{1n} + i_{2n}) \end{aligned}$$

These values substituted in the expression $(v_{13}i_{a1} + v_{23}i_{b2})$ give

$$v_{13}i_{a1} + v_{23}i_{b2} = v_{1n}i_{1n} + v_{2n}i_{2n} + v_{3n}i_{3n} \tag{8a}$$

This last expression is the instantaneous power input to the three phases of the Y, and consequently its average value is equal to the total power input to the load.

Since the average value of the instantaneous products $(v_{13}i_{a1})$ and $(v_{23}i_{b2})$ may be either positive or negative, the deflection of the wattmeter needle may be either to the right or to the left of the zero. Wattmeters, however, are usually designed with a scale reading only to the right. Consequently, when a positive value of the average of the product vi corresponds to a deflection to the right, then the needle goes off the scale to the left when the value of this product is negative. However, by reversing the leads to the potential circuit of the wattmeter, the needle may be made to deflect to the right.

To measure, then, the total three-phase power by the two-wattmeter method, it is necessary to connect the current coils of the two wattmeters in the lines a and b in such a manner that the needles of both meters deflect to the right. It is then necessary to determine whether the two readings shall be added or subtracted. This is determined from the fact that if the average value of the two products $(v_{13}i_{a1})$ and $(v_{23}i_{b2})$ are both of the same sign, then both meters will also read to the right if they are interchanged, keeping the connection to the middle wire c unaltered. If, however, the average values of these two products are of the opposite sign, then when the two meters are interchanged, the

connection to *c* being kept unaltered, the deflection of each instrument will reverse.

Hence, the general rule, *subtract the two readings if, on substituting one meter for the other (the connection to the common wire c being kept unaltered), the deflection reverses; otherwise the two readings are to be added.*

200. Two-wattmeter Method Applied to a Balanced Three-phase Load.—In case the load is balanced and the currents and voltages are both sine waves, the phase relations of the currents and potential drops in the two wattmeters can be readily deduced.

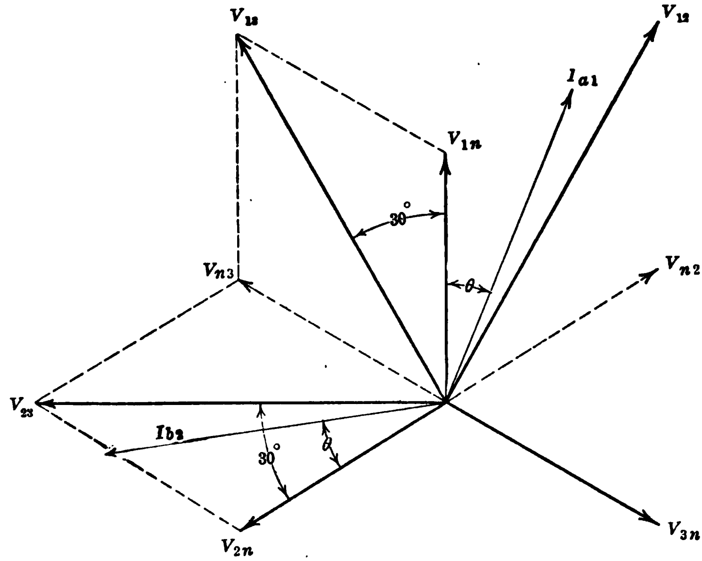


FIG. 161.—Vector diagram for two wattmeters on a balanced three-phase load.

Let V be the r.m.s. value of the line voltage, I the r.m.s. value of the line current, and θ the power-factor angle of the load, taken positive when the current per phase lags the voltage per phase. Then the r.m.s. value of the line currents I_{a1} and I_{b2} is I, and the r.m.s. value of the line voltages V_{13} and V_{23} is V.

Consider the load as a balanced Y. (If it is Δ connected, it may be considered as equivalent to a Y connection, as explained in Article 197.) Then the line current I_{a1} is the same as the current I_{1n} in the phase between terminal No. 1 and the neutral. Similarly, I_{b2} is the same as the current I_{2n} in the phase between terminal No. 2 and the neutral. The vectors representing these

currents then make an angle of 120 degrees with each other, as shown in Fig. 161. The phase voltages for these two phases, viz., the voltages to neutral V_{1n} and V_{2n}, then lead I_{a1} and I_{b2} respectively by the angle θ (see Article 191). The line voltage V_{13} is the vector sum of V_{1n} and V_{n3}, and the line voltage V_{23} is the vector sum of V_{2n} and V_{n3}. The phase angles of these two line voltages with respect to I_{a1} and I_{b2} are then as shown in Fig. 161, viz.,

$$V_{13} \text{ leads } I_{a1} \text{ by } (30° + \theta) \tag{9}$$

$$V_{23} \text{ lags } I_{b2} \text{ by } (30° - \theta) \tag{9a}$$

Whence, noting that the length of each of these two voltage vectors is V (the line voltage) and the length of each of the two current vectors is I (the line current), the reading of wattmeter A (Fig. 160) is

$$P_A = VI \cos (30° + \theta) \tag{10}$$

and the reading of wattmeter B is

$$P_B = VI \cos (30° - \theta) \tag{10a}$$

From these expressions it is evident that when θ lies between $-60°$ and $+60°$ both P_A and P_B are positive. When θ is numerically greater than 60°, and either positive or negative, P_A and P_B are of opposite sign. But $\cos 60° = 0.5$. Hence, the condition that the two wattmeter readings shall be added arithmetically is that the power factor of the load be greater than 50 per cent. When the power factor of the load is less than 50 per cent. (current leading or lagging), the wattmeter readings must therefore be subtracted arithmetically. It must be remembered, however, that this rule applies only when the load is *balanced.*

In general, the power factor of the load is not known. However, in the case of a *balanced* load the following simple test may be used to determine whether the wattmeter readings are to be added or subtracted. Leaving all other connections unchanged, transfer the connection of the potential circuit of meter A, say, from the common wire c to the wire b. Then this meter will read the power corresponding to V_{12} and I_{a1}. But V_{12} is the vector sum of V_{1n} and V_{n2}, and therefore lags I_{a1} by the angle $(30° - \theta)$ (see Fig. 161). Whence the new reading of wattmeter A will be

$$P'_A = VI \cos (30° - \theta) \tag{11}$$

which is the same as the reading of wattmeter B.

Consequently, if when this change in connection is made, the deflection of meter A does not change *in direction*, the sum of the original readings of A and B gives the total power; if the deflection of A reverses, then the difference of the original readings of A and B gives the total power. An inspection of the vector diagram will show that this same rule applies to meter B, when the connection of the potential circuit of B to the common wire c is transferred to the wire a, all other connections remaining unaltered.

Hence the general rule for a *balanced* load; connect the two meters in circuit in such a manner that both deflect to the right and take the two readings P_A and P_B. Then, keeping all other connections unaltered, transfer the connection of the potential coil of one meter from the common wire to the wire in which the current coil of the second meter is connected. If the deflection of the former meter remains in the *same* direction, *add* the original readings of the two meters; if the deflection of this meter reverses, subtract the original readings of the meters.

It is also of interest to note that, in the case of a balanced load, the power-factor angle θ may be expressed directly in terms of the wattmeter readings. For, taking the sum and difference respectively of P_B and P_A, there results

$$P_B + P_A = VI\ [\cos(30° - \theta) + \cos(30° + \theta)] = \sqrt{3}VI\cos\theta$$
$$P_B - P_A = VI\ [\cos(30° - \theta) - \cos(30° + \theta)] = VI\sin\theta$$

Whence, taking the ratio of these two expressions,

$$\tan\theta = \sqrt{3}\,\frac{P_B - P_A}{P_B + P_A} \tag{12}$$

Problem 8.—The two wattmeters connected to a balanced three-phase load, whose power factor is greater than 50 per cent., read 5240 watts and 2380 watts respectively. (*a*) What is the difference in phase between the current in the current coil of the first wattmeter and the potential drop through the potential circuit of the second wattmeter? (*b*) What is the power factor of the load?

Answer.—(*a*) 123 degrees or 57 degrees. (*b*) 83.9 per cent.

Problem 9.—The current coil of a single wattmeter is connected in one of the line wires of a balanced three-phase load of 30 kilowatts. The power factor is 80 per cent., the current lagging. Call the main in which the wattmeter is connected A, the other two line wires B and C respectively. What would the wattmeter read when its potential coil is connected (*a*) between A and B, (*b*) between A and C, and (*c*) between B and C?

Answer.—(*a*) 21.5 kilowatts. (*b*) 8.5 kilowatts. (*c*) 13.0 kilowatts.

Problem 10.—A three-phase Y-connected generator supplies power to a balanced load at a line voltage of 220 volts. The power is measured by the two-wattmeter method, and each wattmeter reads 10 kilowatts. If the resistance and reactance of each phase of the armature is 0.1 ohm and 0.3 ohm respectively, find the generated electromotive force per phase of the alternator.

Answer.—133.2 volts.

Problem 11.—Each phase of a balanced three-phase, Δ-connected load is formed by an impedance in series with a non-inductive resistance of 15 ohms. The voltage across the impedance in each phase is 80, the voltage across the non-inductive resistance in each phase is 150, the resultant voltage across the impedance coil and the non-inductive resistance is 200. What would be the readings of the two wattmeters when connected to this load to measure the power by the two-wattmeter method?

Answer.—3420 watts and 2190 watts.

XVII

SYMBOLIC NOTATION

201. General.—The graphical solution of an alternating-current problem by means of a vector diagram, although it always gives a clear insight into the physical relations involved, is not, as a rule, susceptible of a high degree of accuracy, unless the vector diagram is drawn on a large scale. An algebraic method of expressing the relations given by a vector diagram is therefore frequently desirable. The algebraic method commonly employed is based on the properties of complex numbers, *i.e.*, of numbers which may be expressed as the sum of a real number and a pure imaginary. By a pure imaginary is meant a real number multiplied by the square root of -1.

Although in this method of analysis the symbols employed to represent currents, voltages, impedances, etc., have the *mathematical* form and the *mathematical* properties of complex numbers, these symbols actually represent real physical quantities. This method may therefore be described as the "symbolic" method. If the reader will clearly bear in mind at the outset that it is the *symbols* which have the form of complex numbers, and not the *quantities themselves,* he will avoid much confusion in understanding and applying the method. After one once understands the principles involved, the distribution of alternating currents in a network, no matter how complicated, may be determined as readily as in the case of direct currents

The symbolic method is nothing more than a scheme whereby the *geometrical* relations expressed in a vector diagram may be expressed in *algebraic* form. It is applicable only when the vectors in such a diagram are fixed in length and relative position, that is, when all the alternating quantities represented in the diagram are *sine-wave quantities of the same frequency.* In particular, it is not applicable to transient currents and voltages, for the expressions for such quantities always contain an exponential term When non-sinusoidal, periodic quantities are treated as equivalent sine-wave quantities (see Article 170), the symbolic method is applicable to the same degree of accuracy as the corresponding vector method.

When greater accuracy is needed, each such quantity must be resolved into its fundamentals and harmonics (see Chapter XVIII). When this is done the symbolic method may be applied to *each group* of components of the *same frequency*, and the resultant effect may then be calculated as explained in Chapter XVIII.

Until he becomes thoroughly familiar with the method, the student will find it extremely helpful to draw a vector diagram, not necessarily to scale, showing each of the currents and voltages involved in the particular problem in hand.

202. Symbolic Expression for a Sine-wave Quantity.—As shown in Article 172, a sine-wave quantity may always be rep-

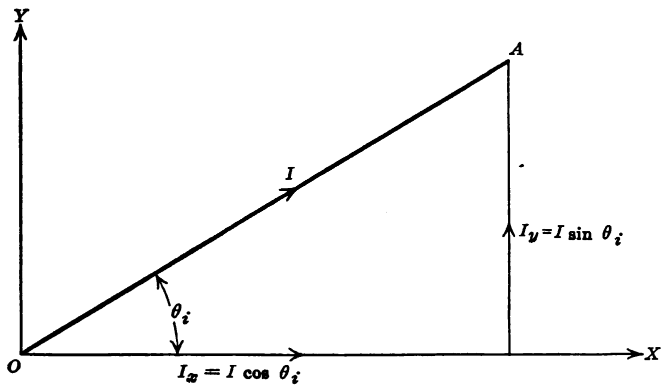


FIG. 162.

resented by a vector $\overline{OA}$ (see Fig. 162) whose length I is equal to the r.m.s. value of this quantity, and which makes with an arbitrarily chosen axis of reference $\overline{OX}$ a constant angle θ_i. Call this axis of reference the X-axis, and let angles be considered positive when measured around in the counter-clockwise direction. The angle θ_i is then to be considered positive when the vector $\overline{OA}$ *leads* the reference line. Let $\overline{OY}$ be a second axis at right angles to, and leading, the reference axis $\overline{OX}$. Call this second axis the Y-axis.

The vector $\overline{OA}$ may be resolved into two components, one parallel to and in the direction of the X-axis, and one parallel to and in the direction of the Y-axis. The component of $\overline{OA}$ in the direction of the X-axis is

$$I_x = I \cos \theta_i \tag{1}$$

and the component of $\overline{OA}$ in the direction of the Y-axis is

$$I_y = I \sin \theta_i \tag{1a}$$

Put

$$\underset{\cdot}{I} = I_x + \sqrt{-1} I_y \tag{2}$$

Note the dot under the I. Then $\underset{\cdot}{I}$, which by *definition* is *algebraically* equal to $I_x + \sqrt{-1}\ I_y$, is called the "symbolic expression" for the sine-wave quantity represented by this vector.

Instead of using the rather cumbersome symbol $\sqrt{-1}$, the letter "j" is usually employed to designate it. That is, putting

$$j = \sqrt{-1} \tag{3}$$

the symbolic expression for the given sine-wave quantity may then be written

$$\underset{\cdot}{I} = I_x + jI_y \tag{4}$$

or

$$\underset{\cdot}{I} = I\ (\cos \theta_i + j \sin \theta_i) \tag{4a}$$

When the given sine-wave quantity is an electric current of r.m.s. value I, equation (4) is then the symbolic expression for this current. Similarly, the symbolic expression for a sine-wave voltage drop of r.m.s. value V is

$$\underset{\cdot}{V} = V_x + jV_y \tag{5}$$

where V_x and V_y are the components, in the directions of the X-axis and Y-axis respectively, of the vector which represents this voltage drop. Or, calling θ_v the angle by which this vector *leads* the axis of reference (the X-axis),

$$\underset{\cdot}{V} = V(\cos \theta_v + j \sin \theta_v) \tag{5a}$$

Again, the symbolic expression for a sine-wave electromotive force of r.m.s. value E is

$$\underset{\cdot}{E} = E_x + jE_y \tag{6}$$

where E_x and E_y are the components, in the directions of the X-axis and Y-axis respectively, of the vector which represents this electromotive force. Or, calling θ_e the angle by which this vector leads the axis of reference (the X-axis),

$$\underset{\cdot}{E} = E(\cos \theta_e + j \sin \theta_e) \tag{6a}$$

As shown in the above expressions, when a single letter is used to designate the symbolic value of a quantity, a *dot* is written under it. This is for the purpose of distinguishing the *symbolic* value from the *numerical* value of the given quantity.

From the above definitions it follows that the relative magnitudes of the two numbers I_x and I_y which enter into the symbolic expression of a sine-wave quantity depend upon the axes of reference chosen. In particular, when axes of reference are so chosen that the vector which represents the given quantity coincides in direction with the X-axis, this vector has no component along the Y-axis; consequently $I_x = I$ and $I_y = 0$, and the symbolic expression for the given quantity, referred to this axis, is

$$\underset{.}{I} = I + j0$$

or simply

$$\underset{.}{I} = I \tag{7}$$

Similarly, when the vector which represents the given quantity coincides in direction with the Y-axis, then the symbolic expression for this quantity, referred to this axis, is

$$\underset{.}{I} = 0 + jI$$

or simply

$$\underset{.}{I} = jI \tag{7a}$$

From equations (1) and (1*a*) it follows that

$$I_x^2 + I_y^2 = I^2 (\cos^2 \theta + \sin^2 \theta) = I^2$$

Whence the r.m.s. value of any sine-wave quantity whose symbolic expression is $(I_x + jI_y)$ is always

$$I = \sqrt{I_x^2 + I_y^2} \tag{8}$$

irrespective of the axis of reference.

Again, from equations (1) and (1*a*)

$$\frac{I_y}{I_x} = \tan \theta_i$$

Whence the angle by which the vector which represents this quantity *leads* the X-axis is

$$\theta_i = \tan^{-1} \left(\frac{I_y}{I_x}\right) \tag{8a}$$

This angle, of course, depends upon the direction chosen as that of the X-axis. For example, were the X-axis so chosen that it

coincides with the vector representing a given voltage, then the angle θ_i given by equation (8*a*) is the angle by which the current $\dot{I} = I_x + jI_y$ *leads* this voltage.

When the vector which represents a given sine-wave quantity *lags* the axis of reference, the angle θ_i is negative, *i.e.*, is expressed by a negative number. Or, calling θ' the angle by which a given vector $\overline{OA}$ (Fig. 163) *lags* the axis of reference, and I the length of this vector, the component of $\overline{OA}$ in the direction of the X-axis is

$$I_x = I \cos \theta' \tag{9}$$

and the component of $\overline{OA}$ *in the direction of* the Y-axis is

$$I_y = -I \sin \theta' \tag{9a}$$

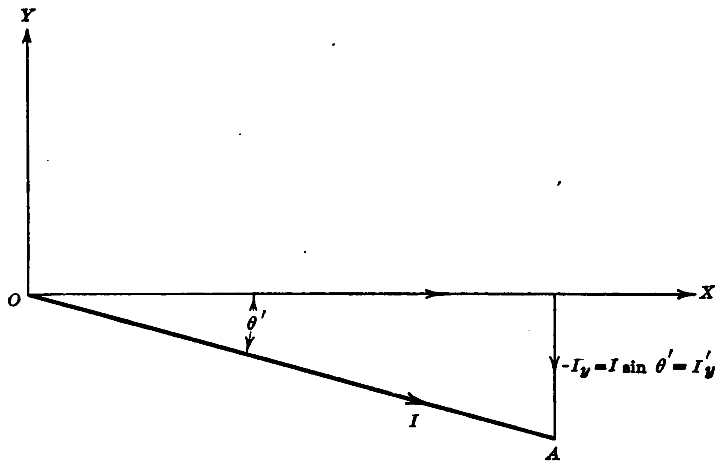


FIG. 163.

The component I_y is negative, since the actual component of $\overline{OA}$ parallel to the Y-axis is in the direction *opposite* to the direction of this axis, *i.e.*, opposite to the directon of the line $\overline{OY}$. Let I'_y be the component of $\overline{OA}$ in the direction *opposite* to that of the Y-axis. Then,

$$I'_y = -I_y$$

or

$$I'_y = I \sin \theta' \tag{9b}$$

The symbolic expression for the quantity represented by the vector $\overline{OA}$ in Fig. 163 is then

$$\dot{I} = I_x + jI_y \tag{10}$$

where I_y is given by equation (9a), or

$$\dot{I} = I_x - jI'_y \tag{10a}$$

where I'_y is given by equation (9b). Both of these expressions are the same as

$$\dot{I} = I(\cos\theta' - j\sin\theta') \tag{10b}$$

Compare with equation (4a).

In general, the algebraic signs to be used before the two terms in the symbolic expression for a sine-wave quantity depend upon the quadrant in which the vector which represents this quantity lies, viz.:

Vector in	Real Term	Imaginary Term
First quadrant	+	+
Second quadrant	−	+
Third quadrant	−	−
Fourth quadrant	+	−

Conversely, the angle by which the vector leads or lags the reference axis depends upon the algebraic signs of the real and imaginary terms in its symbolic expression, viz.:

Real Term	Imaginary Term	Angles
+	+	Between 0° and 90°, leading.
−	+	Between 90° and 180°, leading.
−	−	Between 90° and 180°, lagging.
+	−	Between 0° and 90°, lagging.

Problem 1.—The electromotive force of a given generator is 13,000 volts and leads by 40 degrees the current supplied by this generator. The terminal voltage of the generator is 11,000 volts and leads the current by 30 degrees. The current is 150 amperes. Assume sine-wave currents and voltages.

Taking the vector which represents the current as the axis of reference (the X-axis), write the symbolic expressions for (a) the electromotive force, (b) the terminal voltage and (c) the current.

Answer.—(a) $\dot{E} = 9950 + j8360$. (b) $\dot{V} = 9530 + j5500$. (c) $\dot{I} = 150$.

Problem 2.—If, in Problem 1, the vector which represents the terminal voltage is taken as the axis of reference, what will be the symbolic expres-

sions for (*a*) the electromotive force, (*b*) the terminal voltage and (*c*) the current?

Answer.—(*a*) $\dot{E} = 12{,}800 + j2260$. (*b*) $\dot{V} = 11{,}000 + j0$. (*c*) $\dot{I} = 130 - j75$.

Problem 3.—The symbolic expressions for the two currents in the two branches of a parallel circuit, referred to the voltage drop through this circuit, are

$$\dot{I}_1 = -40 + j60$$
$$\dot{I}_2 = 25 - j15$$

(*a*) What is the numerical (r.m.s.) value of each current? (*b*) What is the difference in phase between each current and the voltage drop? (*c*) What is the difference in phase between the two currents? (*d*) What are the components of the resultant current in phase with, and 90 degrees ahead of, the voltage drop in the external circuit? (*e*) What is the symbolic expression for this current referred to the voltage drop? Compare with the algebraic sum of the symbolic expressions for I_1 and I_2. (*f*) What is the numerical value of this resultant current? (*g*) What is the difference in phase between this resultant current and the voltage drop? (*h*) Draw to scale a complete vector diagram.

Answer.—(*a*) $I_1 = 72.1$ amperes and $I_2 = 29.2$ amperes. (*b*) I_1 *leads* the voltage drop by 123.7 degrees and I_2 lags the voltage drop by 31 degrees. (*c*) 154.7 degrees. (*d*) 15 amperes in the *opposite* direction to the voltage drop, and 45 amperes 90 degrees ahead of the voltage drop. (*e*) $\dot{I} = -15 + j45$, which is the *algebraic sum* of the symbolic expressions for I_1 and I_2, viz., $\dot{I} = \dot{I}_1 + \dot{I}_2$. (*f*) 47.4 amperes. (*g*) The resultant current leads the voltage drop by 108.4 degrees.

Problem 4.—Using the vector which represents the current in Fig. 139, as the axis of reference, show that (*a*) the symbolic expression for the voltage at the receiver is

$$\dot{V} = V(\cos\theta + j\sin\theta)$$

(*b*) The symbolic expression for the impedance drop in the line is

$$\dot{V}_z = rI + jxI = (r + jx)I$$

(*c*) The symbolic expression for the voltage at the generator is

$$\dot{V}_o = (V\cos\theta + rI) + j(V\sin\theta + xI)$$
$$= \dot{V} + (r + jx)I = \dot{V} + \dot{V}_z$$

Note that the *symbolic expression* $\dot{V}_o$ for the generator voltage is the *algebraic* sum of the *symbolic expression* $\dot{V}$ for the voltage at the receiver and the *symbolic expression* $\dot{V}_z$ for the impedance drop in the line. On the other hand, the *numerical value* V_o of the generator voltage is NOT equal to the sum of the *numerical value* V of the receiver voltage and the *numerical value* of the impedance drop V_z, but

$$V_o = \sqrt{(V\cos\theta + rI)^2 + (V\sin\theta + xI)^2}$$

Compare with equation (8).

203. Addition and Subtraction in Symbolic Notation.—As illustrated in the last two problems, the symbolic expression for the resultant of two sine-wave quantities of the same nature (*i.e.* two currents or two voltages), both referred to the same axis of reference, is equal to the *algebraic sum* of the symbolic expressions for these two quantities. This relation is a perfectly general one. That is, let $\underset{\cdot}{I}_1$, $\underset{\cdot}{I}_2$, $\underset{\cdot}{I}_3$, etc., be the symbolic expressions for the several quantities under consideration, *all referred to the same axis of reference.* Then the resultant of these several quantities, referred to this same axis, is

$$\underset{\cdot}{I} = \underset{\cdot}{I}_1 + \underset{\cdot}{I}_2 + \underset{\cdot}{I}_3 +, \text{etc.} \tag{11}$$

Or, using the subscripts x and y, to indicate the algebraic values of the components of these several quantities in the direction of, and 90 degrees ahead of, the axis of reference,

$$I_x + jI_y = (I_{1x} + I_{2x} + I_{3x} + \text{etc.}) + j(I_{1y} + I_{2y} + I_{2y} + \text{etc.}) \tag{11a}$$

This relation is nothing more than a convenient mathematical statement of the fact that the components, in two mutually perpendicular directions, of the resultant of any number of vectors are respectively the sum of the components, in these same directions, of these several vectors (see Article 2). Equation (11*a*) is actually two equations in one, namely, a statement of the fact that

$$I_x = I_{1x} + I_{2x} + I_{3x} + \text{etc.} \tag{12}$$

$$I_y = I_{1y} + I_{2y} + I_{3y} + \text{etc.} \tag{12a}$$

In exactly the same way, when the instantaneous values of a sine-wave quantity I are the *difference* between the instantaneous values of two other sine-wave quantities I_1 and I_2, the symbolic expression for I is

$$I = \underset{\cdot}{I}_1 - \underset{\cdot}{I}_2 \tag{13}$$

where $\underset{\cdot}{I}_1$ and $\underset{\cdot}{I}_2$ are the symbolic expressions for I_1 and I_2. Or, when the subscripts x and y are used to indicate the components of I_1 and I_2 in the direction of, and 90 degrees ahead of, the axis of reference,

$$\underset{\cdot}{I} = I_{1x} - I_{2x} + j(I_{1y} - I_{2y}) \tag{13a}$$

As far as addition and subtraction is concerned, the symbol j may be looked upon as signifying merely that the y-components are 90 degrees ahead of the x-components. The reason for tak-

ing j mathematically equal to $\sqrt{-1}$ is due to the fact that when it is so interpreted, the impedance of a circuit may also be represented by a complex number, and the product of the symbolic impedance of the circuit by the symbolic expression for the current in this circuit gives the symbolic expression for the impedance drop through it (see the next article).

Problem 5.—(*a*) Show that the symbolic expressions for three balanced three-phase currents of r.m.s. value I, when the vector representing one of these currents is taken as the reference vector, are

$$\dot{I}_1 = I + j0$$

$$\dot{I}_2 = \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)I$$

$$\dot{I}_3 = \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)I$$

(*b*) From these expressions prove that the current in the neutral wire of a balanced three-phase Y-connected system is zero. (*c*) Show that if these are the *phase* currents in a Δ-connection, the symbolic expression for the *line* current in the line connected to the junction of phase 2 and phase 3 is $j\sqrt{3}\,I$, and therefore the r.m.s. value of the line current is $\sqrt{3}$ times the r.m.s. value of the phase current, and that this particular line current leads the current in phase 1 by 90 degrees. Compare with Fig. 153.

204. Symbolic Expression for Impedance.—In Fig. 164 the vector $\overline{OA}$ represents a current of r.m.s. value I flowing in circuit whose resistance is r and whose reactance is x. Then the impedance drop V_z in this circuit is represented by the vector $\overline{OB}$, whose component in the direction of $\overline{OA}$ is $\overline{OD} = rI$, and whose component 90 degrees ahead of $\overline{OA}$ is $\overline{DB} = xI$.

Let θ be the angle by which the vector representing the current I leads the reference line. Then the symbolic expression for the current I is

$$\dot{I} = I\ (\cos\theta + j\sin\theta)$$

Or, putting $I_x = I\cos\theta$ and $I_y = I\sin\theta$,

$$\dot{I} = I_x + jI_y \tag{14}$$

The symbolic expression for the impedance drop V_z is (see Fig. 164)

$$V_z = \overline{OF} + j\overline{FB} \tag{15}$$

But

$$\begin{aligned}\overline{OF} &= \overline{OG} - \overline{FG} = \overline{OG} - \overline{CD}\\ &= \overline{OD}\cos\theta - \overline{DB}\sin\theta\end{aligned}$$

and

$$\begin{aligned}\overline{FB} &= \overline{FC} + \overline{CB}\\ &= \overline{OD}\sin\theta + \overline{DB}\cos\theta\end{aligned}$$

But $\overline{OD}$ is equal to the resistance drop rI and $\overline{DB}$ is equal to the reactance drop xI in this circuit. Consequently, equation (15) is equivalent to

$$\underset{\cdot}{V} = (rI\cos\theta - xI\sin\theta) + j(xI\cos\theta + rI\sin\theta)$$

But $I\cos\theta = I_x$ and $I\sin\theta = I_y$. Whence this expression may be written

$$\underset{\cdot}{V} = rI_x - xI_y + j(xI_x + rI_y)$$

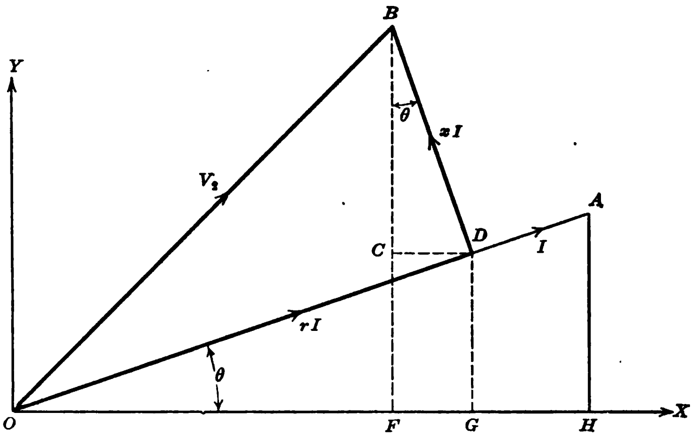


FIG. 164.

This expression is identical with the algebraic product

$$\underset{\cdot}{V} = (r + jx)(I_x + jI_y) \tag{15a}$$

when for j is substituted the numerical value $\sqrt{-1}$, for then $(jx) \times (jI_y) = j^2xI_y = -xI_y$. Note also that $I_x + jI_y = \underset{\cdot}{I}$ is the symbolic expression for the current I.

The complex number

$$\underset{\cdot}{z} = r + jx \tag{16}$$

is *defined* as the *symbolic expression* for the impedance z. Or, when an impedance is expressed in this manner, it is said to be expressed in symbolic notation. Consequently, in symbolic notation, the impedance drop in a circuit may be written

$$\underset{\cdot}{V}_z = \underset{\cdot}{z}\underset{\cdot}{I} \tag{17}$$

where $\underset{\cdot}{z}$ and $\underset{\cdot}{I}$ are respectively the impedance of the circuit and the current in it, both in symbolic notation.

Since in the symbolic expression for an impedance the real term is always the resistance and the imaginary term is always equal numerically to the reactance, *the symbolic expression for an impedance is independent of the axis of reference in the vector diagram.* However, the numerical values of the real and imaginary terms in an impedance *drop* do depend upon the direction of this axis.

A special case of equation (17) was deduced in Problem 4, in which it was shown that the symbolic expression for the impedance drop in a transmission line is $\underset{\cdot}{V}_s = (r + jx)\,\underset{\cdot}{I}$, where the vector representing the current I was taken as the axis of reference.

Equation (17) is identical in mathematical form with Ohm's Law for the resistance drop due to a direct current, viz., with the expression $V_r = rI$, except that in equation (17) the quantities involved are expressed by *complex numbers*, whereas in Ohm's Law all quantities are expressed by real numbers.

Similarly, in symbolic notation the relation between the electromotive force $\underset{\cdot}{E}$ of a generator, its terminal voltage $\underset{\cdot}{V}$, the current $\underset{\cdot}{I}$ through it, and its internal impedance $\underset{\cdot}{z}$, is

$$\underset{\cdot}{V} = \underset{\cdot}{E} - \underset{\cdot}{z}\underset{\cdot}{I} \tag{18}$$

which is of exactly the same form as the corresponding relation $V = E - rI$ for a direct-current generator.

Again, when the circuit under consideration is a receiver of alternating-current energy, the symbolic expression for the terminal voltage is

$$\underset{\cdot}{V} = \underset{\cdot}{E} + \underset{\cdot}{z}\underset{\cdot}{I} \tag{18a}$$

where $\underset{\cdot}{E}$ is the symbolic expression for the electromotive force (if any) in this receiver in the direction *opposite* to that of the current represented by $\underset{\cdot}{I}$, viz., the *back* electromotive force of the receiver. Compare with the corresponding relation $V = E + rI$ for a direct-current receiver.

Problem 6.—A single-phase alternator delivers 25 kilowatts at a power factor of 85 per cent., the current lagging. The terminal voltage of the armature is 500 volts. The armature winding has a resistance of 0.25 ohm and a reactance (assumed constant) of 0.377 ohm. Assume sine-wave currents and voltages.

(*a*) What is the symbolic expression for the impedance of the armature winding? Taking the vector which represents the terminal voltage of the

armature as the axis of reference, write the symbolic expressions for (*b*) the armature terminal voltage, (*c*) the armature current, (*d*) the internal impedance drop, and (*e*) the electromotive force generated in the armature. (*f*) What are the numerical values of these quantities? (*g*) What is the difference in phase between the armature terminal voltage and the generated voltage? Draw a complete vector diagram.

Answer.—(*a*) 0.25 $+j0.377$. (*b*) 500 $+j0$. (*c*) 50 $-j31.0$. (*d*) 24.2 $+ j11.1$. (*e*) 524.2 $+j11.1$. (*f*) Impedance is 0.453 ohm, terminal voltage 500 volts, armature current 58.8 amperes, generated voltage 524.3 volts. (*g*) The generated voltage leads the terminal voltage by 1.2 degrees.

Problem 7.—(*a*) What is the symbolic expression for the internal impedance drop in the generator described in Problem 1, Article 202? (*b*) What is the symbolic expression for the internal impedance of the armature? (*c*) What is the internal resistance and reactance of the armature? (*d*) What is the numerical value of the internal impedance? (*e*) Draw to scale a complete vector diagram, showing the generated voltage, the terminal voltage, the impedance drop, the resistance drop and the reactance drop.

Answer.—(*a*) 420 $+ j2860$. (*b*) $\underset{\cdot}{z} = 2.80 + j19.1$. (*c*) Resistance 2.80 ohms, reactance 19.1 ohms. (*d*) 19.3 ohms.

205. Symbolic Expression for Admittance.—By exactly the same process of reasoning as that employed in the preceding article, it may be shown that the symbolic expression for the current in a circuit, in which there is no externally produced electromotive force, may be written

$$\underset{\cdot}{I} = (g - jb)\underset{\cdot}{V} \tag{19}$$

where $\underset{\cdot}{V}$ is the symbolic expression for the voltage impressed on this circuit, and g and b are respectively the effective conductance and effective susceptance of this circuit. The susceptance b is taken positive when *inductive* (see Article 180).

The complex number

$$\underset{\cdot}{y} = g - jb \tag{20}$$

is therefore taken as the *symbolic expression* for the admittance of the given circuit. Equation (19) may then be written

$$\underset{\cdot}{I} = \underset{\cdot}{y}\underset{\cdot}{V} \tag{21}$$

These expressions should be carefully compared with the corresponding symbolic relations between current, voltage and impedance, as given in the preceding article.

By definition (see Article 180) the *numerical* value of the admittance of a circuit is equal to the reciprocal of the *numerical* value of its impedance. It may readily be shown that this

same relation also holds between the *symbolic* expressions for admittance and impedance, when for j is substituted the numerical value $\sqrt{-1}$, and therefore $j^2 = -1$. That is, if $\dot{z} = r + jx$ is the symbolic expression for an impedance of resistance r and reactance x, then the symbolic expression for the admittance corresponding to this impedance is

$$\dot{y} = \frac{1}{\dot{z}} \tag{22}$$

This follows from the fact that

$$\frac{1}{r + jx} = \frac{r - jx}{(r + jx)(r - jx)} = \frac{r - jx}{r^2 + x^2} = \frac{r}{r^2 + x^2} - j\left(\frac{x}{r^2 + x^2}\right)$$

From equations (29a) and (29b) of Article 180, the conductance and susceptance corresponding to a resistance r and reactance x in series are respectively

$$g = \frac{r}{r^2 + x^2} \qquad \text{and} \qquad b = \frac{x}{r^2 + x^2}$$

Whence

$$\frac{1}{r + jx} = g - jb \tag{22a}$$

In like manner it may be shown that

$$\frac{1}{\dot{y}} = \dot{z} \tag{23}$$

Problem 8.—A certain condenser has an effective leakage conductance of 1.5 micromhos, and a capacity of 0.02 microfarads.

(a) What is the symbolic expression for the admittance of this condenser to a 60-cycle voltage? (b) What is the symbolic expression for the current taken by this condenser when a sine-wave voltage of 25,000 volts is impressed on its terminals, taking the vector which represents the voltage as the axis of reference? (c) What is the numerical value of this current? (d) The power factor of the condenser? (e) Draw to scale a complete vector diagram.

Answer.—(a) $\dot{y} = (1.5 + j7.54) \times 10^{-6}$. ($b$) $\dot{I} = 0.0375 + j0.188$. ($c$) 0.192 ampere. ($d$) 19.5 per cent., current leading.

Problem 9.—A sine-wave voltage of 200 volts impressed on the terminals of a certain circuit, in which there is no externally produced electromotive force, produces a sine-wave current of 50 amperes which lags this voltage by 60 degrees.

(a) Write the symbolic expression for the voltage in the circuit referred to the current. (b) Write the symbolic expression for the current referred to itself. (c) From these two expressions, what is the symbolic expression for the impedance of the circuit? (d) In a similar manner, using the

voltage as the reference vector, write the symbolic expression for the admittance of the circuit. (*e*) If the frequency is 60 cycles per second, what is the effective inductance of this circuit?

Answer.—(*a*) $\dot{V} = 100 + j173$. (*b*) $\dot{I} = 50$. (*c*) $\dot{z} = 2 + j3.46$. (*d*) $\dot{y} = 0.125 - j0.217$. (*e*) 9.18 millihenry.

206. Symbolic Expressions for Impedances in Series and in Parallel.—As shown in Article 181, two or more impedances z_1, z_2, etc., in series are equivalent to a single impedance

$$Z = \sqrt{R^2 + X^2}$$

where

$$R = r_1 + r_2 + \text{etc.} \tag{24}$$
$$X = x_1 + x_2 + \text{etc.} \tag{24a}$$

and r_1, r_2, etc., are the resistances, and x_1, x_2, etc., are the reactances of these several impedances. From Article 204, the symbolic expression for the impedance Z is $\dot{Z} = R + jX$, and the symbolic expressions for the individual impedances are $\dot{z}_1 = r_1 + jx_1$, $\dot{z}_2 = r_2 + jx_2$, etc. Whence, from (24) and (24*a*),

$$\begin{aligned}\dot{Z} &= (r_1 + r_2 + \text{etc.}) + j(x_1 + x_2 + \text{etc.})\\ &= (r_1 + jx_1) + (r_2 + jx_2) + \text{etc.}\end{aligned}$$

or,

$$\dot{Z} = \dot{z}_1 + \dot{z}_2 + \text{etc.} \tag{25}$$

That is, when *impedances* are expressed in *symbolic notation*, they may be added algebraically, just as resistances in a direct-current circuit are added algebraically.

In a similar manner it may be shown that when *admittances* are expressed in *symbolic notation* they may also be added algebraically. That is, when two or more circuits are connected in parallel, and there is no externally produced electromotive force in any one of these circuits, these several circuits are equivalent to a single circuit whose admittance in symbolic notation is

$$\dot{Y} = \dot{y}_1 + \dot{y}_2 + \text{etc.} \tag{26}$$

where $\dot{y}$, $\dot{y}_2$, etc., are the symbolic expressions for the admittances of the several circuits.

When any one of the parallel circuits contains an externally produced electromotive force, *e.g.*, when any one of the circuits is the armature winding of a motor or generator, or either winding of a transformer, equation (26) is not applicable. Such circuits must be treated in the same manner as direct-current cir-

cuits which contain electromotive forces (see Article 51). For example, equations (16), (17), (18), (18a) and (19) of Article 51 are directly applicable to two alternating-current generators in parallel, provided the resistances are changed to impedances, and all quantities are expressed in *symbolic notation*. However, it should be carefully noted that the equations for *power* cannot be thus treated; for example, equation (20) of Article 51 for the power lost in the windings of two direct-current generators is not applicable to alternating-current generators.

Equation (26) may also be written

$$\frac{1}{\dot{Z}} = \frac{1}{\dot{z}_1} + \frac{1}{\dot{z}_2} + \text{etc.} \tag{26a}$$

where $\dot{Z}$ is the symbolic expression for the single impedance equivalent to the several impedances in parallel, and $\dot{z}_1, \dot{z}_2$, etc., are the symbolic expressions for the individual impedances. Compare with equation (12), Article 51, for resistances in parallel in direct-current circuits.

A special case of frequent occurrence is that of only two impedances in parallel. In this case equation (26a) may be written

$$\dot{Z} = \frac{\dot{z}_1\dot{z}_2}{\dot{z}_1 + \dot{z}_2} \tag{26b}$$

Compare with equation (14), Article 51.

When the impedances $\dot{z}_1$ and $\dot{z}_2$ in equation (26b) are expressed in terms of their resistances and reactances, this equation becomes

$$R + jX = \frac{(r_1 + jx_1)(r_2 + jx_2)}{(r_1 + jx_1) + (r_2 + jx_2)}$$

or

$$R + jX = \frac{r_1r_2 - x_1x_2 + j(r_1x_2 + r_2x_1)}{(r_1 + r_2) + j(x_1 + x_2)} \tag{26c}$$

The second term of this last expression is of the general form

$$\frac{A + jB}{C + jD} \tag{27}$$

where A, B, C, and D are all real quantities. Such an expression may be readily reduced to the form $M + jN$ by multiplying

numerator and denominator by $C - jD$, remembering that $j^2 = -1$. This gives

$$\frac{A + jB}{C + jD} = \frac{(A + jB)(C - jD)}{C^2 + D^2}$$
$$= \frac{AC + BD}{C^2 + D^2} + j\left(\frac{BC - AD}{C^2 + D^2}\right)$$

Whence the numerical value of the real part of (27) is

$$M = \frac{AC + BD}{C^2 + D^2} \qquad (27a)$$

and the numerical value of the imaginary part is

$$N = \frac{BC - AD}{C^2 + D^2} \qquad (27b)$$

A fraction of the form $\frac{A + jB}{C + jD}$ is called an "irrational" complex number. By the process just described such an expression may always be "rationalized," or reduced to the form of a simple complex number, viz., to the form $M + jN$, where M and N are real numbers. As may be readily seen by rationalizing equation (26c), this process usually leads to a relatively complex expression, when algebraic symbols are used for the numbers involved. However, when actual numbers are used, and each arithmetical operation is performed as it arises, the process of rationalizing a complex fraction is relatively simple.

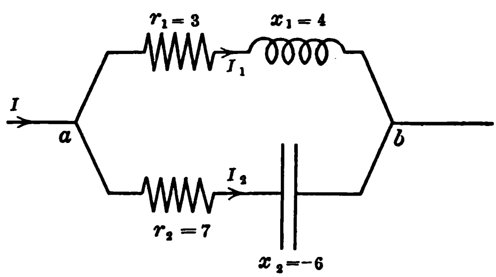


FIG. 165.

For example, referring to Fig. 165, if the first impedance has a resistance of 3 ohms and an inductive reactance of 4 ohms, and the second impedance has a resistance of 7 ohms and a capacity reactance of 6 ohms, the symbolic expressions for these impedances are

$$\dot{z}_1 = 3 + j4$$
$$\dot{z}_2 = 7 - j6$$

Whence

$$\dot{z}_1 + \dot{z}_2 = 10 - j2$$

$$\dot{z}_1\dot{z}_2 = (3 + j4)\ (7 - j6)$$

$$= 21 + 24 + j(28 - 18) = 45 + j10$$

$$\frac{\dot{z}_1\dot{z}_2}{\dot{z}_1 + \dot{z}_2} = \frac{45 + j10}{10 - j2} = \frac{(45 + j10)(10 + j2)}{100 + 4}$$

$$= \frac{450 - 20 + j(90 + 100)}{104}$$

$$= \frac{430}{104} + j\frac{190}{104}$$

$$= 4.13 + j1.83$$

The two impedances in parallel are then equivalent to a single impedance whose resistance is 4.13 ohms and whose reactance is 1.83 ohms. The reader should check this deduction by the method given in Article 183.

Problem 10.—Solve Problem 13, Article 183, by the symbolic method.

Problem 11.—Three impedances A, B and C have resistances of 5, 8 and 3 ohms respectively and reactances of 14, 0 and -10 ohms respectively. Find the symbolic expressions of the resultant impedance of these three impedances, (*a*) when they are connected in series, (*b*) when they are connected in parallel. (*c*) Find the symbolic expression of the resultant admittance of these three impedances when they are connected in parallel.

Answer.—(*a*) $16 + j4$. (*b*) $5.56 - j0.902$. (*c*) $0.1751 + j0.0284$.

Problem 12.—Two impedances A and B, of 4 and 12 ohms resistance respectively, and of 15 and -8 ohms reactance respectively, are connected in series. If the current established in this circuit is 10 amperes, write the symbolic expression for the potential drop, (*a*) through A, (*b*) through B, and (*c*) through the entire circuit, all referred to the current as the axis of reference. (*d*) What are the numerical values of these voltages?

Answer.—(*a*) $40 + j150$. (*b*) $120 - j80$. (*c*) $160 + j70$. (*d*) 155, 144, 175 volts respectively.

Problem 13.—Two impedances A and B, whose resistances are 4 ohms and 5 ohms respectively, and whose reactances are respectively 8 ohms inductive and 3 ohms condensive, are connected in parallel. If the total current supplied to this parallel circuit is 20 amperes, write the symbolic expressions for (*a*) the currents in A and B, and (*b*) the potential drop across the parallel circuit, all referred to the total current supplied to the circuit. (*c*) What are the numerical values of these currents and what is the numerical value of the voltage?

Answer.—(*a*) $5.66 - j9.81$ and $14.34 + j9.81$. (*b*) $101.1 + j6.0$. (*c*) 11.3 and 17.4 amperes and 101.3 volts respectively.

Problem 14.—An impedance A is connected in series with two impedances B and C connected in parallel. The resistances of A, B and C are 3, 5 and 6 ohms respectively and the reactances are 5, -7 and 3 ohms respectively. 200 volts are impressed across the entire circuit. Write the symbolic expressions for the currents in A, B and C, referred to the potential drop A.

Answer.—$11.4 - j19.1$; $12.5 - j2.80$; $-1.04 - j16.3$.

207. Kirchhoff's Laws in Symbolic Notation.—The distribution of direct currents in any kind of conducting network, no matter how complicated, may always be determined by the application of the two fundamental principles known as Kirchhoff's Laws, viz., (1) the algebraic sum of the currents entering each junction point of the network is zero, and (2) the algebraic sum of the resistance drops around any closed loop in the network is equal to the algebraic sum of the electromotive forces in this loop. From the definitions and relations stated in the preceding articles it follows that, on the assumption that the alternating currents and electromotive forces in a given network are all *sinusoidal*, these same two laws also hold for the *symbolic expressions* for these quantities, except that the impedance drops (in symbolic notation) must be substituted for the resistance drops.

That is, for sine-wave currents and voltages:

(1) The algebraic sum of the symbolic expressions for the several currents entering any junction point in a conducting network is zero, *provided* the same axis of reference is used for all these currents.

(2) The algebraic sum of the symbolic expressions for the impedance drops around any closed loop in the network is equal to the algebraic sum of the symbolic expressions of the electromotive forces acting around this loop in the same direction, *provided* the same axis of reference is used for all the currents and all the electromotive forces.

These two laws may be stated mathematically as follows: Let $\dot{I}_{10}$, $\dot{I}_{20}$, etc., be the symbolic expressions for the currents which *enter* any junction point O (see Fig. 166), *all referred to the same axis of reference.* Then

$$\dot{I}_{10} + \dot{I}_{20} + \text{etc.} = 0 \tag{28}$$

Let $\dot{z}_{12}$, $\dot{z}_{23}$, etc., be the symbolic expressions for the impedances of the separate branches which form any closed loop 1, 2, 3, etc.,

in the network (see Fig. 167). Let $\dot{I}_{12}$, $\dot{I}_{23}$, etc., be the symbolic expressions for the currents which exist in these several branches in the same relative direction *around* the network, and all referred to the same axis of reference. Let $\dot{E}_{12}$, $\dot{E}_{23}$, etc., be the symbolic expressions for the electromotive forces (if any) in these branches in the *same* relative direction *around* the network as the currents,

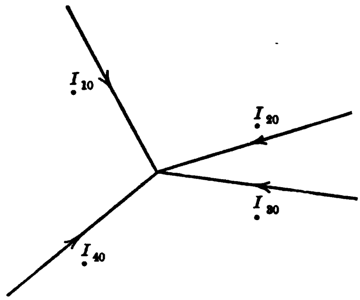


FIG. 166.

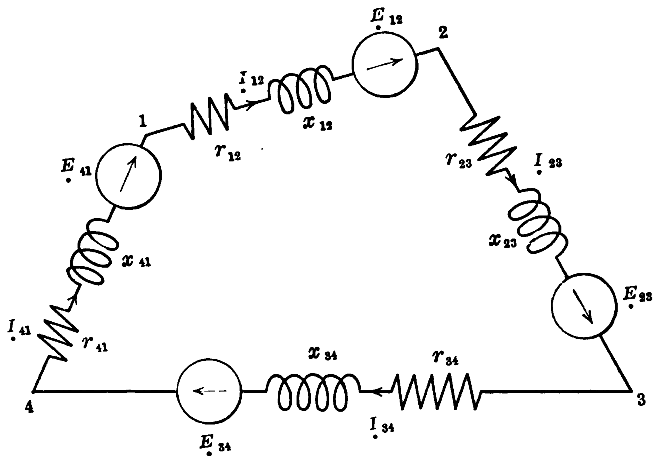


FIG. 167.

and all referred to the same axis of reference as used for the currents. Then

$$\dot{z}_{12}\dot{I}_{12} + \dot{z}_{23}\dot{I}_{23} + \text{etc.} = \dot{E}_{12} + \dot{E}_{23} + \text{etc.} \tag{29}$$

The electromotive forces represented by the E's in this equation are the *externally* induced electromotive forces, such as those due to generators or motors, or to the mutual inductance of the two windings of a transformer. The electromotive forces due to

self-inductance and capacity are taken account of by the reactance terms of the impedances.

In applying Kirchhoff's Laws in symbolic notation, careful attention must be paid to algebraic signs. The double subscript notation will be found a great help in this connection, particularly when the network is at all complicated. When the double subscript notation is used, an electromotive force, say, from 1 to 2 is represented by the symbol $\dot{E}_{12}$, whereas an electromotive force from 2 to 1 is represented by the symbol $\dot{E}_{21}$. If $\dot{E}_{12}$ referred to a given axis of reference is $\dot{E}_{12} = E_x + jE_y$, then $\dot{E}_{21}$ referred to this same axis of reference is $\dot{E}_{21} = -E_x - jE_y$. Hence, substracting $\dot{E}_{12}$ is the same as adding $\dot{E}_{21}$.

Another way of indicating the direction which is to be taken as the positive sense of the quantity represented by a given symbol is to place an arrow on the circuit diagram, directly along side the given symbol, and take as the positive sense of the given quantity the direction of this arrow. When this scheme is employed but a single subscript is needed. In simple networks this method is often preferable to the double subscript method.

An impedance, of course, has no direction, and when written with a double subscript merely signifies the impedance *between* the two points corresponding to these subscripts.

Each equation of the form (28) or (29) is in reality equivalent to two equations, since the sum of all the real terms on one side must be equal to the sum of all the real terms on the other side, and similarly, the sum of all the imaginary terms on one side must be equal to the sum of all the imaginary terms on the other side; all fractions having been "rationalized" as explained in Article 206. That is, the component, in the direction of the X-axis of the *resultant* of any number of vectors must be equal to the algebraic sum of the components, in this direction, of all the individual vectors; and similarly for the component along the Y-axis. These two laws therefore enable one to calculate both components of every voltage and of every current, when the impedances and the electromotive forces are known.

It should be clearly borne in mind that the above equations are true only for the *symbolic expressions* for the currents, electromotive forces and impedances. These equations are not true for the numerical values of these quantities. That is, *the sum of the currents entering a junction as measured by ammeters in*

the various branches is not necessarily zero; again, *the sum of the potential drops in the various branches of a closed loop, as measured by voltmeters connected across the various branches, is not necessarily zero.*

Again, these equations hold only when the currents and voltages vary *sinusoidally* with time, have the *same* frequency, and when *the resistances and reactances are constant.* When, however, the resistances, inductances and capacities are constant, it may be shown that a similar set of equations holds for *each* frequency that may be present. Since the equations are all linear in the *I's* and *E's*, the currents and electromotive forces of any given frequency will be uninfluenced by the presence of currents or electromotive forces of any other frequency. Hence, when the harmonics present in each electromotive force are known (see Article 213), the harmonics present in each current may be calculated by solving the equations corresponding to the frequency of this particular harmonic, these equations being exactly the same as would hold were all the other harmonics absent.

Note particularly that the above equations do not hold for transient currents.

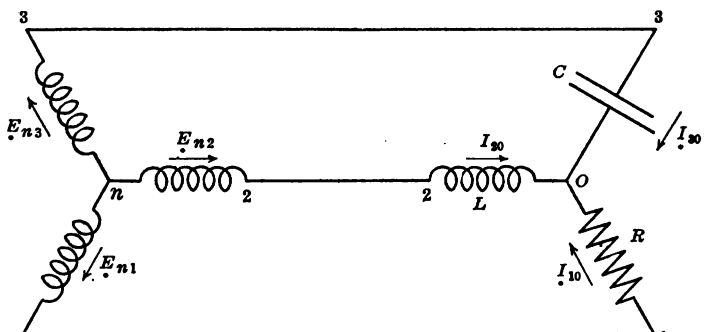


FIG. 168.

Problem 15.—A non-inductive resistance of 10 ohms, a coil whose reactance is 10 ohms and whose resistance is negligible, and a condenser whose capacity reactance is 10 ohms and whose effective conductance is zero, are connected in *Y* (see Fig. 168), and this *Y* is connected to the terminals of a *Y*-connected generator whose electromotive forces are balanced. The internal impedance of the generator is negligible. The electromotive force generated in each phase of the generator is 100 volts.

(*a*) What are the impedances of the three branches of the *Y*-connected load? (*b*) Taking the vector which represents the electromotive force from the neutral to terminal No. 1 as the axis of reference, write the symbolic

expressions for the three generated electromotive forces. (c) Designating the symbolic values of the currents in the three branches of the load $\dot{I}_{10}$, $\dot{I}_{20}$ and $\dot{I}_{30}$, write in symbolic notation the loop equations for the loop $n-1-1-0-2-2-n$ and for the loop $n-3-3-0-2-2-n$. (d) Write the junction point equation at the neutral 0. (e) Solve these three equations for $\dot{I}_{10}$, $\dot{I}_{20}$ and $\dot{I}_{30}$. (f) What are the r.m.s. values of the currents in the noninductive resistance, in the coil and in the condenser? (g) Are these currents balanced? (h) What is the difference of potential between the neutral point of the generator and the neutral point of the load. (i) Draw to scale a complete vector diagram showing the currents in the three branches of the Y and the voltages between each terminal and its neutral point.

Answer:

(a) $\dot{z}_{10} = 10 + j0$
$\dot{z}_{20} = 0 + j10$
$\dot{z}_{30} = 0 - j10$

(b) $\dot{E}_{n1} = 100 + j0$
$\dot{E}_{n2} = -50 + j86.6$
$\dot{E}_{n3} = -50 - j86.6$

(c) $150 - j86.6 = 10\dot{I}_{10} - j10\dot{I}_{20}$
$-j173.2 = -j10\dot{I}_{30} - j10\dot{I}_{30}$

(d) $\dot{I}_{10} + \dot{I}_{20} + \dot{I}_{30} = 0$

(e) $\dot{I}_{10} = -17.32$
$\dot{I}_{20} = 8.66 + j32.32$
$\dot{I}_{30} = 8.66 - j32.32.$

(f) 17.32 amperes in the resistance and 33.4 amperes in both the coil and the condenser. (g) No. (h) 273.2 volts.

208. Power and the Real and Imaginary Terms in the Symbolic Expressions for Current and Voltage.—The average power input (or output) corresponding to a current I and voltage V may be readily expressed in terms of the real and imaginary terms in their symbolic expressions. Referring to Fig. 169, let $\overline{OA}$ and $\overline{OB}$ be respectively the vectors which represent the current in any given circuit and the voltage drop through this circuit in the direction of the current. Let I_x be the components of $\overline{OA}$ in the direction of the X-axis (axis of reference) and let I_y be the component of $\overline{OA}$ in the direction of the Y-axis. Let V_x and V_y be the corresponding components of $\overline{OB}$. Let θ_i and θ_v be the angles by which $\overline{OA}$ and $\overline{OB}$ *lead* the reference axis. Let $\theta = \theta_v - \theta_i$ be the difference in phase between the voltage and the current.

The average power input to the circuit is then, from Article 167,

$$\begin{aligned} P &= VI \cos \theta \\ &= VI \cos (\theta_v - \theta_i) \\ &= VI \cos \theta_v \cos \theta_i + VI \sin \theta_v \sin \theta_i \end{aligned}$$

From the figure

$$\begin{aligned} I\cos\theta_i &= \overline{OC} = I_x \\ I\sin\theta_i &= \overline{CA} = I_y \\ V\cos\theta_v &= \overline{OD} = V_x \\ V\sin\theta_v &= DB = V_y \end{aligned}$$

Consequently, when the symbolic expressions for the current and voltage, referred to *any* common axis are

$$\dot{I} = I_x + jI_y \tag{30}$$

and

$$\dot{V} = V_x + jV_y \tag{30a}$$

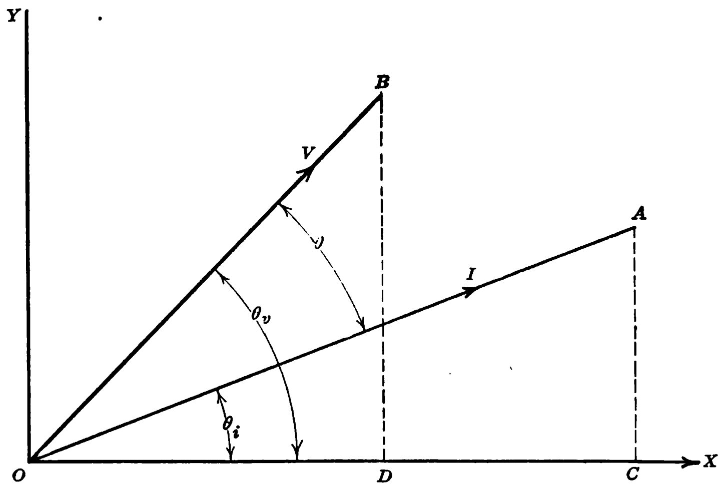


FIG. 169.

the average power input to the circuit is

$$P = V_xI_x + V_yI_y \tag{31}$$

In applying this expression attention must be paid to the algebraic signs of the X and Y components. For example, consider a current and voltage whose symbolic expressions are

$$\begin{aligned} \dot{I} &= 5 - j10 \\ \dot{V} &= 100 + j40 \end{aligned}$$

The component of the current in the direction of the Y-axis is then negative, namely, $I_y = -10$. Whence the power input is

$$P = 5 \times 100 - 10 \times 40 = 100$$

It should be carefully noted that the above expression, for **power,** equation (31), **is NOT the real part of the product of the symbolic expressions for the current and voltage.** The product of $\underset{\cdot}{V} = V_x + jV_y$ and $I = I_x + jI_y$ is, noting that $j^2 = -1$,

$$\underset{\cdot}{V}\underset{\cdot}{I} = V_xI_x - V_yI_y + j(V_xI_y + V_yI_x)$$

The real part of this product is

$$V_xI_x - V_yI_y$$

whereas the average power is

$$P = V_xI_x + V_yI_y$$

A convenient way of remembering the expression for power, however, is that it is equal to the *real part of the product* $\underset{\cdot}{V}\underset{\cdot}{I}$ *with the sign between the two terms REVERSED.*

Problem 16.—A 25-cycle, three-phase, balanced load of 20,000 kilowatts is delivered to a substation over a three-wire line 50 miles long. The voltage between wires at the substation is 60,000 volts, and the substation power factor is 90 per cent., the current lagging. The line wires are No. 0000 A. W. G. stranded copper symmetrically spaced 6 feet apart. The resistance of each line wire is 0.27 ohm per mile and the reactance is 0.30 ohm per mile. Sine-wave currents and voltages are to be assumed. (This is the same load and line as in Problem 3, Article 191 and in Problem 6, Article 196.)

Using the vector which represents the line current in one wire as the axis of reference, write the symbolic expressions (*a*) for this line current, (*b*) for the voltage at the load between this line and the neutral, (*c*) for the impedance of this line, (*d*) for the impedance drop in this line, and (*e*) for the voltage at the power house between this line and the neutral. (*f*) What is the r.m.s. value of the voltage to neutral at the power house? (*g*) What is the r.m.s. value of the line voltage at the power house? (*h*) What is the power factor of the total load on the power house? Using equation (31), calculate (*i*) the power input of each phase of the load, (*j*) the power lost in each line wire, and (*k*) the power output of each phase of the power house. Compare with the solution to Problems 3 and 6, Articles 191 and 196.

Answer:—(*a*) $\underset{\cdot}{I} = 214$. (*b*) $\underset{\cdot}{V} = 31{,}180 + j15{,}100$. (*c*) $\underset{\cdot}{z} = 13.5 + j15$. (*d*) $\underset{\cdot}{z}\underset{\cdot}{I} = 2890 + j3210$. (*e*) $\underset{\cdot}{V}_o = 34{,}070 + 18{,}310$. (*f*) 38,700 volts. (*g*) 67,000 volts. (*h*) 88.1 per cent. (*i*) 214 × 31,180 watts = 6670 kilowatts. (*j*) 214 × 2890 watts = 619 kilowatts. (*k*) 214 × 34,070 watts = 7290 kilowatts.

Problem 17.—Referring to Problem 16, what is the symbolic expression for (*a*) the line current referred to the voltage to neutral at the load, and for (*b*) the voltage to neutral at the power house referred to this same axis? (*c*) From these two symbolic expressions calculate the power output of each phase of the power house. Compare with (*k*) of Problem 16.

Answer.—(*a*) $\underset{\cdot}{I} = 192.6 - j93.3$. (*b*) $38{,}640 + j1630$. (*c*) 38,640 × 192.6 − 1630 × 93.3 watts = 7287 kilowatts, which checks with (*k*), Problem 16.

XVIII

NON-SINUSOIDAL ALTERNATING QUANTITIES

209. Introduction.—As has been noted a number of times, the wave shapes of alternating quantities (*e.g.* currents, voltages, magnetic fluxes, etc.) are seldom pure sine waves. The analysis of any phenomenon involving non-sinusoidal quantities is based upon a principle first stated by Fourier, namely, that any continuous alternating quantity of constant frequency may be resolved into a sum of sine-wave quantities whose frequencies are integer multiples of this frequency. That is, when the instantaneous values i of a given alternating quantity of constant frequency f vary with time in any manner whatever, this variation may always be expressed by a series of the form

$$i = I_1 \sin (\omega t + \theta_1) + I_2 \sin (2\omega t + \theta_2) + I_3 \sin (3\omega t + \theta_3) + \text{etc.} \tag{1}$$

where $\omega = 2\pi f$, and I_1, I_2, I_3, etc., and $\theta_1, \theta_2, \theta_3$, etc., are constants. A series of the form given by equation (1) is called a "Fourier's series."

The first term in this expression, which is a sine-wave quantity of the same frequency as the given alternating quantity, is called the "fundamental," or "first harmonic," of the given quantity. The term which has twice the frequency of the fundamental is called the "second harmonic," the term which has three times the frequency of the fundamental is called the "third harmonic," etc. The sum of the harmonics (including the fundamental or first harmonic) may be conveniently designated the "resultant." In Fig. 170 is shown, by the heavy curve, the wave form of a non-sinusoidal alternating quantity which contains in addition to the fundamental, the third and seventh harmonics.

When the wave form of a non-sinusoidal quantity is symmetrical, in such a manner that every pair of ordinates a half period apart are numerically equal, but opposite in sign, only the odd harmonics can be present. This may be seen by plotting an even harmonic of any order (*e.g.*, a harmonic whose frequency is 2, or 4, or 6 times the frequency of the fundamental).

It will then be evident that any two ordinates in this harmonic, at a distance apart equal to a half period of the resultant, will have the same sign. Hence the corresponding ordinates of the resultant will be unequal. Consequently, if each positive ordinate of the resultant is numerically equal to the negative ordinate which is a half period from it, the resultant cannot contain an even harmonic.

In practice the wave shapes of most alternating quantities are symmetrical in the sense just described, and therefore contain only the odd harmonics. This, however, is not always the case, for under certain special conditions even harmonics may be produced.

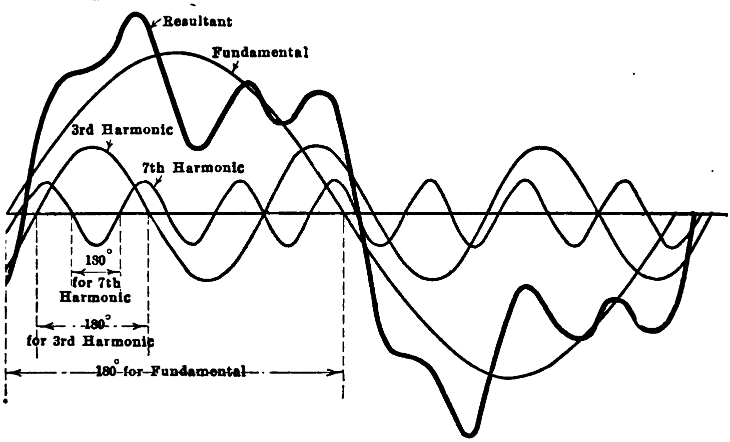


FIG. 170.

The wave shape of the electromotive force of an alternator is determined primarily by the distribution of the magnetic flux in its air-gap. If the flux density in the gap varies sinusoidally with distance measured circumferentially around the armature, the electromotive force will be sinusoidal, assuming the armature conductors to be symmetrically spaced. Such a distribution of flux can be approximated by properly shaping the field poles, but can never be completely realized in a slotted armature. The effect of the teeth is to produce harmonics of relative high order, as evidenced by the sharp jags in the resultant wave.

As will be shown in Article 214, it is always possible to determine, from the wave shape of the resultant, the order, maximum values and phase angles of all the harmonics present.

210. The Oscillograph.—The wave shape of a current or voltage may be determined experimentally by means of an instrument called an "oscillograph." This instrument is essentially a galvanometer whose moving element, called the vibrator, has an extremely short period, usually about $1/6000$ of a second. A vibrator of such short period will follow with practical accuracy any variations in the current through it which has a frequency of less than 2000 cycles per second.

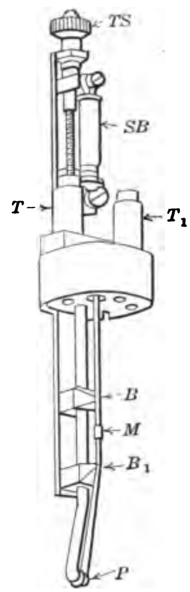


FIG. 171.—Vibrating element of oscillograph.

The vibrator (Fig. 171) consists of two strips of flattened wire stretched over bridges, with a small mirror cemented to the strips. This arrangement is equivalent to a coil of one turn. This coil is placed between the poles of a magnet, thereby forming a one-turn galvanometer. To provide suitable damping, the vibrator is immersed in oil.

To determine the wave shape of the current in a given circuit, the vibrator is connected across a non-inductive shunt in series with the circuit. A strong light, usually from an arc-lamp, is projected on the mirror, from which it is reflected, through suitable prisms and lenses, to a ground-glass plate or photographic film. For direct visual observations a rotating mirror is placed in the path of the light reflected from the mirror, so that the

spot of light on the ground-glass plate is given a uniform motion at right angles to its deflection produced by the oscillation of the vibrator. When a photographic film is used the film itself is moved, and the rotating mirror is not needed. Due to the two motions at right angles to each other, the spot of light reflected from the mirror will trace on the ground-glass plate, or on the photographic film, a curve whose ordinates are directly proportional to the instantaneous values of the current, which curve, by definition, is the wave shape of the current.

The wave shape of a voltage is determined in exactly the same manner, except that the vibrator, in series with a high resistance, is shunted across the terminals of the circuit. The oscillograph then becomes an "instantaneous voltmeter."

The curves obtained by means of an oscillograph are called "oscillograms." For further details in regard to the construction and use of this instrument see the article on *Oscillographs* in Pender's *Handbook for Electrical Engineers.*

211 Power Corresponding to a Non-harmonic Alternating Current and Voltage.—Let the equations for the current and voltage be

$$i = \sqrt{2}\,[I_1 \sin(\omega t + \theta_1) + I_2 \sin(2\omega t + \theta_2) + I_3 \sin(3\omega t + \theta_3)] + \text{etc.} \qquad (2)$$

$$v = \sqrt{2}\,[V_1 \sin(\omega t + \theta'_1) + V_2 \sin(2\omega t + \theta'_2) + V_3 \sin(3\omega t + \theta'_3)] + \text{etc.} \qquad (2a)$$

The power *at any instant* is always the product of the current and voltage at this instant, viz., $p = vi$. This product is equal to the product of the second members of equations (2) and (2a). Carrying out this multiplication term by term, and making use of the relation expressed by equation (13), Article 167, there results

$$\begin{aligned} p = vi = {} & V_1I_1 \cos(\theta_1 - \theta'_1) + V_2I_2 \cos(\theta_2 - \theta'_2) + V_3I_3 \cos(\theta_3 - \theta'_3) \\ & - V_1I_1 \cos(2\omega t + \theta_1 + \theta'_1) - V_2I_2 \cos(4\omega t + \theta_2 + \theta'_2) \\ & - V_3I_3 \cos(6\omega t + \theta_3 + \theta'_3) + V_2I_1 \cos(\omega t + \theta'_2 - \theta_1) \\ & + V_3I_1 \cos(2\omega t + \theta'_3 - \theta_1) + V_1I_2 \cos(\omega t + \theta_2 - \theta'_1) \\ & + V_3I_2 \cos(\omega t + \theta'_3 - \theta_2) + V_1I_3 \cos(2\omega t + \theta_3 - \theta'_1) \\ & + V_2I_3 \cos(\omega t + \theta_3 - \theta'_2) - V_2I_1 \cos(3\omega t + \theta_1 + \theta'_2) \\ & - V_3I_1 \cos(4\omega t + \theta_1 + \theta'_3) - V_1I_2 \cos(3\omega t + \theta_2 + \theta'_1) \\ & - V_3I_2 \cos(5\omega t + \theta_2 + \theta'_3) - V_1I_3 \cos(4\omega t + \theta_3 + \theta'_1) \\ & - V_2I_3 \cos(5\omega t + \theta_3 + \theta'_2) + \text{etc.} \end{aligned}$$

The *average* power P for a complete cycle of the resultant (which is equal to a complete cycle of the fundamental) is equal to the average value of this expression between the limits $\omega t = 0$ and $\omega t = 2\pi$. Over such a period the average of each term which contains the variable t is zero (see Article 167). Hence the average power is

$$P = V_1 I_1 \cos (\theta_1 - \theta'_1) + V_2 I_2 \cos (\theta_2 - \theta'_2) + V_3 I_3 \cos (\theta_3 - \theta'_3) + \text{etc.} \quad (3)$$

where the V's and I's are the r.m.s. values of the harmonics.

Therefore, when there exists in a circuit an alternating current and an alternating voltage of any kind whatever, the average power is equal to the sum of the values of the average power corresponding to *each pair of harmonics of the same frequency*. Consequently, the average power input corresponding to each pair of harmonics of the *same* frequency is independent of what other harmonics may be present. Note particularly that when any harmonic is absent, in *either* the voltage or the current, this harmonic contributes nothing to the average power.

212. R.m.s. Value of a Non-sinusoidal Alternating Quantity.— By definition, the square of the r.m.s. value of any kind of alternating quantity is the *average* (or mean) of the squares of its instantaneous values. Let the instantaneous values of the given quantity be expressed by the Fourier's series

$$i = \sqrt{2}[I_1 \sin (\omega t + \theta_1) + I_2 \sin (2\omega t + \theta_2) + I_3 \sin (3\,\omega t + \theta_3)]$$

Then the square of its instantaneous value is of exactly the same form as the power corresponding to a current and voltage each identically equal to this expression. Consequently, from equation (3), the mean of the squares of the instantaneous values of i is

$$I^2 = I_1^2 + I_2^2 + I_3^2 + \text{etc.}$$

Hence the r.m.s. value of the given quantity is

$$I = \sqrt{I_1^2 + I_2^2 + I_3^2 + \text{etc.}} \quad (4)$$

That is, the r.m.s. value of any non-sinusoidal alternating quantity is equal to the square root of the sum of the squares of the r.m.s. values of all the harmonics (including the fundamental) which it contains. For example, when the r.m.s. values of the harmonics in a given current wave are I_1, I_2, I_3, etc., the r.m.s. value of this current is given by equation (4). Again, when the

r.m.s. values of the harmonics in a given electromotive force wave are E_1, E_2, E_3, etc., the r.m.s. value of this electromotive force is

$$E = \sqrt{E_1{}^2 + E_2{}^2 + E_3{}^2 + \text{etc.}} \tag{4a}$$

Problem 1.—The equation of the current in a circuit is

$$i = 50 \sin (\omega t + 20°) + 30 \sin (3\omega t - 15°) + 10 \sin (5\omega t + 30°)$$

and the equation of the potential drop through this circuit is

$$v = 100 \sin (\omega t - 10°) + 40 \sin (3\omega t - 30°)$$

Determine (*a*) the r.m.s. value of the current, (*b*) the r.m.s. value of the potential difference across the circuit, (*c*) the average power absorbed by the circuit, (*d*) the power factor of the circuit, and (*e*) the equivalent phase difference between the sine-wave current and voltage "equivalent to" the actual current and voltage.

Answer.—(*a*) 41.8 amperes. (*b*) 76.1 volts. (*c*) 2744 watts. (*d*) 86.3 per cent. (*e*) 30.3 degrees.

213. Currents Produced by Non-sinusoidal Voltages.—Consider first the particular case of a coil of resistance r and constant self-inductance L. Irrespective of the wave shape of the current, the voltage drop through this coil at any instant is (see Article 126)

$$v = ri + L\frac{di}{dt}$$

Let the current be the sum of two sine-wave currents of any two *different* frequencies f_1 and f_2, that is, let

$$i = i_1 + i_2$$

where

$$i_1 = \sqrt{2}\, I_1 \sin (2\pi f_1 t - \alpha_1)$$
$$i_2 = \sqrt{2}\, I_2 \sin (2\pi f_2 t - \alpha_2)$$

Then the above expression for v may be written

$$v = v_1 + v_2$$

where

$$v_1 = ri_1 + L\frac{di_1}{dt} = \sqrt{2}(rI_1) \sin (2\pi f_1 t - \alpha_1) + \sqrt{2}(2\pi f_1 L I_1) \cos (2\pi f_1 t - \alpha_1)$$

$$v_2 = ri_2 + L\frac{di_2}{dt} = \sqrt{2}\, (rI_2) \sin (2\pi f_2 t - \alpha_2) + \sqrt{2}\, (2\pi f_2 L I_2) \cos (2\pi f_2 t - \alpha_2)$$

From Article 175, the r.m.s. values of v_1 and v_2 are then

$$V_1 = I_1\sqrt{r^2 + (2\pi f_1 L)^2}$$
$$V_2 = I_2\sqrt{r^2 + (2\pi f_2 L)^2}$$

and the angles by which these voltage drops lead the currents i_1 and i_2 are θ_1 and θ_2 where

$$\cos\theta_1 = \frac{r_1}{\sqrt{r^2 + (2\pi f_1 L)^2}}$$

$$\cos\theta_2 = \frac{r_2}{\sqrt{r^2 + (2\pi f_2 L)^2}}.$$

From Article 211, the power input to the coil due to the resultant current is

$$P = V_1 I_1 \cos\theta_1 + V_2 I_2 \cos\theta_2$$
$$= r\,(I_1^2 + I_2^2) = rI^2$$

where $I = \sqrt{I_1^2 + I_2^2}$ is the r.m.s. value of this resultant current Consequently, the power dissipated in a conductor by two sine-wave currents of *different* frequencies, flowing simultaneously in this conductor, is the sum of the heat losses due to each current separately.

Note that this relation does not hold for two currents of the *same* frequency, for the r.m.s. value of the resultant of two such currents is not $\sqrt{I_1^2 + I_2}$, but is $\sqrt{I_1^2 + I_2^2 + 2I_1I_2\cos\theta}$, where θ is the difference in phase between these two currents.

In an exactly similar manner it may be shown that in any circuit whose resistance, inductance and capacity are *constants*, the effects produced by sine-wave currents and voltages of any *one* frequency are in no way influenced by the presences of electromotive forces and currents of any *other* frequency. From this fundamental principle it follows that in any network of *constant* resistances, inductances and capacities, the effects produced by non-sinusoidal alternating electromotive forces may always be determined by considering each frequency separately, and applying thereto the relations developed in the preceding chapters for pure sine-wave currents and voltages. The r.m.s. value of the resultant current or voltage in any branch is then the square root of the sum of the squares of the r.m.s. values of its sine-wave components.

The total power input is the sum of the power inputs for the separate harmonics (see Article 211).

It should be carefully noted that this principle is applicable only when the resistance, the inductance, and the electrostatic capacity of each branch of the network is *constant*. When any one of these factors varies with time, its variation will in general

introduce additional harmonics in the currents and voltages. Cases in which such variations occur are a coil with an iron core (variable inductance), an electric arc (variable resistance) and a condenser in the dielectric of which the phenomenon of electric absorption is pronounced (variable capacity).

As an illustration of the relations above developed, consider a coil which has an inductance of 0.01 henry and therefore a reactance of $2\pi \times 60 \times 0.01 = 3.77$ ohms to a 60-cycle *sine-wave* voltage. Let a non-sinusoidal 60-cycle voltage containing the third harmonic be impressed on this coil. The reactance of the coil to the third harmonic is then $2\pi \times 3 \times 60 \times 0.01 = 11.31$ ohms, *i.e.*, *three* times its reactance to the fundamental. Assume the resistance of the coil to be negligible in comparison with its 60-cycle reactance. Let the r.m.s. value of the *fundamental* of the impressed voltage be 100 volts, and the r.m.s. value of the *third harmonic* be 20 volts. Then r.m.s. value of the fundamental of the current will be

$$I_1 = \frac{100}{3.77} = 26.5 \qquad \text{amperes}$$

and the r.m.s. value of the third harmonic in the current will be

$$I_3 = \frac{20}{11.37} = 1.7 \qquad \text{amperes}$$

Comparing the relative values of the third harmonic in the voltage and the third harmonic in the current, it will be noted that the third harmonic in the voltage is 20 per cent. of the fundamental, whereas the third harmonic in the current is only $\frac{1.7}{26.5}$ $= 6.7$ per cent. of the fundamental of the current. Hence the inductance of a circuit tends to damp out the harmonics in the current produced by a non-sinusoidal voltage, the degree of damping being greater the higher the order of the harmonic. That is, an inductance tends to "smooth out" the wave shape of the current.

On the other hand, a condenser in a circuit tends to distort the current wave. This follows from the fact that the reactance of a condenser varies *inversely* as the frequency (see Article 179). Hence, a third harmonic in the impressed voltage equal to 20 per cent. of the fundamental of this voltage, will produce a third harmonic in the charging current equal to 60 per cent. of the fundamental of this current. On the other

hand, by using a coil and condenser *in parallel*, with their inductance and capacity of such values as to produce resonance (see Article 184) for a given harmonic, this harmonic can be practically eliminated from the current wave.

Problem 2.—A 60-cycle single-phase generator and a 25-cycle single-phase generator are connected in series. The r.m.s. values of the electromotive forces of the two generators are respectively 500 volts and 100 volts. Each electromotive force is a pure sine wave.

(*a*) What would be the reading of a voltmeter connected across the free terminals of the two machines? (*b*) Were a circuit consisting of a resistance of 10 ohms, an inductance of 0.1 henry, and a capacity of 100 microfarads, all in series, connected to these terminals, what would an ammeter in this circuit read? The internal impedance of each generator is to be neglected. (*c*) What would be the total power input to the circuit? Were the frequencies of both electromotive forces 60 cycles per second and were these electromotive forces in phase, what would be (*d*) the reading of the voltmeter, (*e*) the reading of the ammeter, and (*f*) the total power input?

Answer.—(*a*) 510 volts. (*b*) 33.4 amperes. (*c*) 11.14 kilowatts. (*e*) 600 volts. (*f*) 40 amperes. (*g*) 16.0 kilowatts.

Problem 3.—A 25-cycle, three-phase Δ-connected generator develops an electromotive force per phase of 1000 volts r.m.s. value. This electromotive force contains the third harmonic, and the r.m.s. value of this harmonic is 10 per cent. of the r.m.s. value of the *resultant* electromotive force. The resistance per coil of the generator is 0.1 ohm and the reactance per coil at 25 cycles is 0.3 ohm. Assume the inductance to be constant.

(*a*) What will be the value of the phase current when the generator is supplying no external load? (*b*) Will this current be a sine-wave current, and if so, what will be its frequency? (*c*) Plot to the same scale of time, the resultant and third harmonic of the electromotive force in *each* of the three phases. (*d*) What will be the wave shape and r.m.s. value of the terminal voltage of the generator (between wires) at no load?

Answer.—(*a*) 110.4 amperes. (*b*) Sine-wave current with frequency of 75 cycles. (*d*) Pure sine wave, having an r.m.s. value of 995 volts.

Problem 4.—(*a*) Prove that in Y-connected three-phase generator there can be no third harmonic in the line voltage, even though a third harmonic is present in the phase voltage. (*b*) Prove that when the neutral points of a Y-connected generator and a Y-connected load are connected by a wire, a third harmonic, if present in the phase voltage of the generator, will produce a triple-frequency current in the neutral wire.

214. Determination of the Maximum Value and Phase Angle of the Harmonics in a Wave of Any Shape.—When the wave shape of an alternating quantity is known, the r.m.s. value and the phase angle of the harmonic of any order may be readily determined. Let

$$i = I_1 \sin x + I_2 \sin (2x + \theta_2) + I_3 \sin (3x + \theta_3) + \text{etc.} \qquad (5)$$

be the Fourier's series which represents the given wave. In this expression $x = 2\pi ft$, where f is the frequency. The problem is to determine the constants I_1, I_2, I_3, etc., and θ_2, θ_3, etc., which will make the curve represented by (5) coincide with the actual wave.

Consider, for example, the third harmonic. Multiply equation (5) by $\sin 3x$, and integrate with respect to x over an entire period of the wave, *i.e.*, between the limits $x = o$ and $x = 2\pi$. This gives

$$\int_o^{2\pi} i \sin 3x dx = \int_o^{2\pi} [I_1 \sin x \sin 3x + I_2 \sin (2x + \theta_2) \sin 3x + I_3 \sin (3x + \theta_3) \sin 3x + \text{etc.}]\, dx$$

But the integral of each term in the right-hand member of this equation between the limits 0 and 2π is zero, except for the particular term $I \sin (3x + \theta) \sin 3x$, the integral of which from 0 to 2π is $\frac{2\pi I_3}{2} \cos \theta_3 = \pi I_3 \cos \theta_3$ (see Article 167). Hence

$$\int_o^{2\pi} i \sin 3x\, dx = \pi I_3 \cos \theta_3$$

The integral $\int_o^{2\pi} i \sin 3x dx$ may be determined graphically by plotting the expression $i \sin 3x$ as ordinates against x as abscissas, and determining the area of this curve by means of a planimeter. Call this area A_3, then

$$A_3 = \pi I_3 \cos \theta_3 \tag{6}$$

Next, multiply the equation (5) by $\cos 3x = \sin\left(3x + \frac{\pi}{2}\right)$. In exactly the same manner as before

$$\int_o^{2\pi} i \cos 3x dx = \pi I_3 \cos \left(\frac{\pi}{2} - \theta_3\right) = \pi I_3 \sin \theta_3$$

and the value of $\int_o^{2\pi} i \cos 3x dx$ may be determined graphically by plotting the curve $i \cos 3x$ and finding its area by means of a planimeter. Call this area B_3, then

$$B_3 = \pi I_3 \sin \theta_3 \tag{7}$$

From equations (6) and (7) it follows that

$$I_3 = \frac{1}{\pi}\sqrt{A_3^2 + B_3^2} \tag{8}$$

and

$$\theta_3 = \tan^{-1}\left(\frac{B_3}{A_3}\right) \tag{8a}$$

Note that in case the wave is symmetrical in the manner described in Article 209, it is unnecessary to look for the *even* harmonics. Also, when the wave is symmetrical, the curves $i \sin 3x$ and $i \cos 3x$ need be plotted for only a *half* period of the wave. Calling α_3 and β_3 the areas of these curves for half a period of the wave, the values of I_3 and θ_3 may be written

$$I_3 = \frac{2}{\pi}\sqrt{\alpha_3^2 + \beta_3^2} \tag{9}$$

$$\theta_3 = \tan^{-1}\left(\frac{\beta_3}{\alpha_3}\right) \tag{9a}$$

This method of determining the harmonics present in the wave is of course applicable to the determination of the harmonic of any order. It is, however, extremely tedious in its application (except in certain simple cases). A more practical method for analyzing a wave is given in the next article.

As an example of the method of analysis here given, consider the case of a rectangular wave, shown in Fig. 172. Let I be the numerical value of current, which is constant during each half cycle.

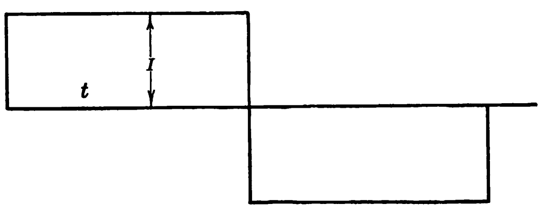


FIG. 172.

Since the wave is symmetrical, only the odd harmonics can be present. Consider the nth harmonic, where n is any *odd* positive integer. Then

$$\alpha_n = \int_0^\pi I \sin nx dx = \left(-\frac{I}{n}\cos nx\right)_0^\pi = \frac{2I}{n}$$

and

$$\beta_n = \int_0^\pi I \cos nx dx = \left(\frac{I}{n}\sin nx\right)_0^\pi = 0$$

Hence from equation (9) and (9*a*)

$$I_n = \frac{2}{\pi}\sqrt{\left(\frac{2I}{n}\right)^2 + (0)^2} = \frac{4I}{\pi n}$$

and

$$\theta_n = \tan\frac{0}{\frac{2I}{n}} = 0$$

Hence all the *odd* harmonics exist in a symmetrical rectangular wave, the maximum values of the harmonics varying inversely as their order n. A symmetrical rectangular wave having a maximum value I may then be represented by the infinite series

$$i = \frac{4I}{\pi}\sum_{n=1}^{n=\infty}\frac{\sin\,(2\pi nft)}{n} \tag{10}$$

where n has all the *odd* values from 1 to ∞, and f is the frequency of the wave.

215. The Fisher-Hinnen Method for Analyzing a Non-Sinusoidal Wave.—(See *Electric Journal*, Vol. 5, p. 386 and *Elektrotechnische Zeitschrift*, May 9, 1901.) This method is much simpler than that described above, except in the rare cases where the resultant wave may be represented by a simple integrable function. The method is based on the following facts:

1. The *algebraic* sum of any n equally spaced ordinates of a *sine* wave, when these ordinates are spaced $\frac{m}{n}$th of a wave length apart, where m is any integer which is *not* a multiple of n, is zero.

2. The algebraic sum of n ordinates of a *sine* wave when these ordinates are spaced $\frac{m}{n}$ wave lengths apart, *where m is a multiple of n*, is equal to n times the ordinate of this wave at any one of these points.

3. The maximum ordinate of any *sine* wave is equal to the square root of the sum of the squares of any two ordinates spaced a quarter of a wave length apart.

4. Let y_1 be the ordinate of a *sine* wave at any point at an angular distance x_1 from the origin, and let y_2 be another ordinate of this wave a quarter of a wave length to the *right* of x_1. Then the angular distance measured from the point x_1 toward the *left*

to the point at which this wave first crosses the x-axis in the positive direction is

$$\phi = \tan^{-1}\frac{y_1}{y_2}$$

Consider a wave of any form whatever (for example, the resultant wave shown in Fig. 170) and let the highest harmonic of this wave be the nth. Let $y_1, y_3, y_5, \ldots y_{2n-1}$ be n ordinates of this wave spaced $\frac{360}{n}$ degrees apart, where 360 degrees corresponds to a complete wave length of the given wave. Let a'_1, a'_3, a'_5, be the corresponding ordinates of the fundamental or first harmonic of this wave, a''_1, a''_3, a''_5, be the corresponding ordinates of the second harmonic of this wave, a'''_1, a'''_3, a'''_5, etc., be the corresponding ordinates of the third harmonic, and so on; the corresponding ordinates of the nth harmonic being $a_1^{(n)}, a_3^{(n)}, a_5^{(n)}$, etc. Then

$$\begin{aligned}
y_1 &= a'_1 + a''_1 + a'''_1 + \ldots\ldots\ldots\ldots a_1^{(n)} \\
y_3 &= a'_3 + a''_3 + a'''_3 + \ldots\ldots\ldots\ldots a_3^{(n)} \\
y_5 &= a'_5 + a''_5 + a'''_5 + \ldots\ldots\ldots\ldots a_5^{(n)} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
y_{2n-1} &= a'_{2n-1} + a''_{2n-1} + a'''_{2n-1} + \ \ldots\ldots\ldots a_{2n-1}^{(n)}
\end{aligned}$$

The ordinates a'_1 to a'_{2n-1} of the first harmonic are ordinates of a *sine* wave and are spaced $\frac{1}{n}$th of a wave apart, and therefore, from Proposition 1, their sum is zero. Similarly, the ordinates a''_1 to a''_{2n-1} are ordinates of a *sine* wave of *half* the wave length of the fundamental and therefore the angular distance between these ordinates is $\frac{2}{n}$th of the wave length of the *sine* wave of which they are the ordinates; hence the sum of the ordinates a''_1 to a''_{2n-1} is zero. Similarly for all the other ordinates except those of the nth harmonic. The latter are spaced $\frac{n}{n} = 1$ wave length apart, and therefore, from Proposition 2, their sum is equal to n times the value of the ordinate of this harmonic at any one of the points 1, 3, 5, etc. Hence the value of the ordinate of the nth harmonic at the point 1 is

$$A_n = \frac{1}{n}(y_1 + y_3 + y_5 + \ldots\ldots\ldots\ldots y_{2n-1}) \qquad (11)$$

Similarly, if $y_2, y_4, y_6, \ldots y_{2n}$ are ordinates of the given wave one quarter of a wave length of the nth harmonic to the

right of the first set of ordinates, their sum will be equal to n times the ordinate of the nth harmonic a quarter of a wave length of this harmonic from y_1; call this ordinate B_n, then

$$B_n = \frac{1}{n}(y_2 + y_4 + y_6 + \ldots\ldots\ldots\ldots y_{2n}) \qquad (11a)$$

Then from Proposition 3, the maximum value of the ordinate of the nth harmonic is

$$Y_n = \sqrt{A_n^2 + B_n^2} \qquad (12)$$

From Proposition 4, the angular distance to the *left* of y_1 at which this harmonic cuts the X-axis in the positive direction is

$$\phi'_n = \tan^{-1}\frac{A_n}{B_n} \qquad (13)$$

when 360 degrees are taken equivalent to a wave length of the nth harmonic. When 360 degrees are taken equivalent to a wave length of the *given wave* this angular distance is

$$\phi_n = \frac{1}{n}\tan^{-1}\frac{A_n}{B_n} \qquad (13a)$$

Consider next the mth harmonic, and erect two sets of m ordinates, the ordinates of each set being spaced $\frac{360}{m}$ degrees apart (considering a wave length of the given wave as equivalent to 360 degrees) and the second set a quarter of a wave length of this harmonic to the right of the first set. Then, if the harmonics of higher order are not multiples of m, the ordinate of the mth harmonic at the point 1 is

$$A_m = \frac{1}{m}(y_1 + y_3 + y_5 \ldots\ldots\ldots\ldots y_{2m-1}) \qquad (14)$$

and the ordinate of the mth harmonic at the point 2, which is a quarter wave length of this harmonic to the right of 1, is

$$B_m = \frac{1}{m}(y_2 + y_4 + y_6 + \ldots\ldots\ldots\ldots y_{2m}) \qquad (14a)$$

Whence the maximum value of this harmonic is

$$Y_m = \sqrt{A_m^2 + B_m^2} \qquad (15)$$

and it cuts the X-axis at the angular distance

$$\phi_m = \frac{1}{m}\tan^{-1}\frac{A_m}{B_m} \qquad (16)$$

to the *left* of the first ordinate, when 360 degrees are taken equivalent to the wave length of the original wave.

If there also exists in the given wave a harmonic of the nth order, where n is a multiple of m, that is if $n = km$, where k is an integer, then from Proposition 2, since each set of these m ordinates is spaced $\frac{n}{m} = k$ wave lengths of the nth harmonic apart, the sum of the first set of m ordinates also contains m times the ordinate of the nth harmonic at the point 1. Calling A'_n the ordinate of the nth harmonic at the point 1, then

$$A_m = \frac{1}{m}(y_1 + y_3 + y_5 + \dots\dots\dots\dots + y_{2n-1}) - A'_n \qquad (17)$$

Similarly,

$$B_m = \frac{1}{m}(y_2 + y_4 + y_6 + \dots\dots\dots\dots y_{2n}) - B'_n \qquad (17a)$$

where B'_n is the ordinate of the nth harmonic at the point 2.

A similar correction must be applied for all other harmonics of higher order than the mth, if the orders of these harmonics are multiples of m.

When only the fundamental, third, fifth and seventh harmonics are present, the procedure becomes relatively simple. Under these conditions there are no correction terms, except to the fundamental, for 3, 5, and 7 are not multiplies of one another. Also it is necessary to consider the ordinates of a half wave only. To determine the third harmonic, divide the base line of this half wave into $2n = 6$ equal parts, and measure the ordinates at the beginning of each of these six segments. Let these ordinates be $y_1, y_2, \ldots y_6$. Let the beginning of the first segment be taken where the given wave cuts the X-axis, and call this point the origin, then $y_1 = 0$ and therefore

$$A_3 = \frac{1}{3}(y_5 - y_3)$$

$$B_3 = \frac{1}{3}(y_2 + y_6 - y_4)$$

The maximum value of the third harmonic is then

$$Y_3 = \sqrt{A_3^2 + B_3^2}$$

and it cuts the X-axis at the angular distance

$$\phi_3 = \frac{1}{3}\tan^{-1}\left(\frac{A_3}{B_3}\right)$$

to the *left* of the origin, taking the base of the given half-wave as 180 degrees. The expression for the third harmonic is then

$$Y_3 \sin 3(x + \phi_3)$$

Similarly, starting at the same point and dividing the half wave into $2n = 10$ segments,

$$A_5 = \frac{1}{5}(y_5 + y_9 - y_3 - y_7)$$

$$B_5 = \frac{1}{5}(y_2 + y_6 + y_{10} - y_4 - y_8)$$

$$Y_5 = \sqrt{A_5^2 + B_5^2}$$

$$\phi_5 = \frac{1}{5}\tan^{-1}\left(\frac{A_5}{B_5}\right)$$

and the expression for this harmonic is

$$Y_5 \sin 5(x + \phi_5)$$

The determination of the seventh harmonic is carried out by an exactly similar process, dividing the base line into $2n = 14$ equal parts.

The ordinates of the fundamental at the origin and at the point corresponding to one quarter of a wave length from the origin are then respectively

$$A_1 = -A_3 - A_5 - A_7$$

and

$$B_1 = y_m + B_3 - B_5 + B_7$$

where y_m is the ordinate of the given wave corresponding to the point a quarter of a wave length from the origin (*i.e.*, the mid-ordinate of the given wave), and A_3, A_5, A_7, B_3, B_5, and B_7 are the quantities above determined. The maximum value of the fundamental is then

$$Y_1 = \sqrt{A_1^2 + B_1^2}$$

and it cuts the X-axis at the angular distance

$$\phi_1 = \tan^{-1}\left(\frac{A_1}{B_1}\right)$$

to the left of the origin. Its expression is therefore,

$$Y_1 \sin (x + \phi_1)$$

The equation of the given wave is then

$$y = Y_1 \sin (x + \phi_1) + Y_3 \sin 3(x + \phi_3) + Y_5 \sin 5(x + \phi_5) + Y_7 \sin 7(x + \phi_7) \tag{18}$$

The r.m.s. value of the given wave is

$$Y = \sqrt{\frac{Y_1^2 + Y_3^2 + Y_5^2 + Y_7^2}{2}} \tag{18a}$$

Multiplying both sides of equation (18) by dx and integrating between the limits $x = 0$ and $x = \pi$, it may readily be shown that the average value of the given wave is

$$Y_{\text{aver.}} = \frac{2}{\pi}\left[Y_1 \cos \phi_1 + \frac{1}{3} Y_3 \cos 3\,\phi_3 + \frac{1}{5} Y_5 \cos 5\,\phi_5 + \frac{1}{7} Y_7 \cos 7\phi_7\right] \tag{18b}$$

In employing the above formulas strict attention must be paid to algebraic signs.

Problem 5.—Six equally spaced ordinates (30 degrees apart) in one-half of a symmetrical wave have the values

$y_1 = 0$	$y_4 = 940$
$y_2 = 676$	$y_5 = 1004$
$y_3 = 660$	$y_6 = 554$

Ten equally spaced ordinates (18 degrees apart) in this wave have the values

$y_1 = 0$	$y_3 = 719$	$y_5 = 702$	$y_7 = 1086$	$y_9 = 639$
$y_2 = 470$	$y_4 = 678$	$y_6 = 940$	$y_8 = 920$	$y_{10} = 375$

(*a*) Plot this half wave on cross-section paper, taking 1 inch equal to 30 degrees, and 1 inch equal to an ordinate of 200. (*b*) What is the maximum value and phase angle of the third harmonic (calling the base of a half wave of the *resultant* wave 180 degrees)? (*c*) What is the maximum value and phase angle of the fifth harmonic (calling the base of a half-wave of the *resultant* wave 180 degrees)? (*d*) What is the maximum value and phase angle of the fundamental (calling the base of the *resultant* half-wave 180 degrees)? (*e*) What is the r.m.s. value of the resultant wave? (*f*) What is the average value of the resultant wave? (*g*) What is its form factor? (*h*) Its crest factor? (*i*) Plot to scale on the same sheet as the original curve the fundamental and each harmonic, add graphically, and compare the resultant thus found with the original curve. (*j*) Write the equation of the resultant wave.

Answer.—(*b*) Maximum value 150, phase angle 16.6 degrees measured to the *left* of the origin. (*c*) Maximum value 100, phase angle 13.6 degrees measured to the *right* of the origin. (*d*) Maximum value 1000, phase angle 1.25 degrees to the *right* of the origin. (*e*) 718. (*f*) 663. (*g*) 1.083. (*h*) 1.53. (*j*) $y = 1000 \sin (x - 1.25°) + 150 \sin 3(x + 16.6°) + 100 \sin 5(x - 13.6°)$.

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